$H\infty$ control system design attenuating initial state uncertainties: Evaluation by a magnetic suspension system

H_∞ Control System Design Attenuating Initial State Uncertainties :

Evaluation by a Magnetic Suspension System

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Abstract

This paper deals with an H_∞ control attenuating initial-state uncertainties, and its application to a magnetic suspension system. An H_∞ control problem, which treats a mixed attenuation of disturbance and initial-state uncertainty for linear time-invariant systems in the infinite-horizon case, is examined. The mixed attenuation supplies H_∞ controls with good transients or assures H_∞ controls of robustness against initial-state uncertainty. We apply this method to a magnetic suspension system, and evaluate attenuation property of the proposed disturbance and initial-state uncertainty via simulations and experiments.

keywords: H_∞ Control, DIA Control, Initial-State Uncertainties, Magnetic Suspension Systems

1 Introduction

Mixed attenuations of disturbances and initial state uncertainties are expected to supply H_∞ control problem with some good transient properties. In the finite-horizon case, a generalized type of H_∞ control problem which formulated and solved by Uchida and Fujita[1] and Khargonekar et al.[2]. This problem was extended to the infinite-horizon case, and a generalized type of H_∞ control problem which considers a mixed attenuation of disturbance and initial-state uncertainty in the infinite-horizon case was derived by Uchida and authors[3]. In this paper, we evaluate the effectiveness of the proposed approach[3] with a magnetic suspension system via simulations and experiments. A magnetic suspension system can suspend a magnetic body by magnetic forces without any contact[4]. Feedback control, especially robust feedback control is indispensable for a magnetic suspension system, which is essentially an unstable system. Recently, this seems to be one of hot topic in control application field. First, we show that the proposed controller has a relatively better transient property than the conventional standard H_∞ controller. Next, a role of the weighting matrix N for the initial state x_0 is shown via numerical simulation. N is a measure of relative importance of the initial-state uncertainty attenuation to the disturbance attenuation. Finally, usefulness and effectiveness of the free parameter Ψ of the mixed attenuation of disturbance and initial-state uncertainty is examined via experiments.

2 Mixed attenuation of disturbance and initial-state uncertainties

Consider the linear time-invariant system which is defined on the time interval $[0, \infty)$ and described by

$$\begin{aligned} \dot{x} &= Ax + Bu + Dv, \quad x(0) = x_0 \\ y &= Cx + w \\ z &= Fx \end{aligned} \tag{1}$$

where $x \in R^n$ is the state and x_0 is the initial state; $u \in R^r$ is the control input; $y \in R^m$ is the observed output; $g := (z' \;\; u')' \in R^{q+r}$ is the controlled output; $h := (v' \;\; w')' \in R^{p+m}$ is the disturbance. Without loss of generality, we regard x_0 as the initial-state uncertainty, and $x_0 = 0$ as known initial-state case. Each element of the disturbance $h(t)$ is a square integrable function defined on $[0, \infty)$; A, B, C, D and F are constant matrices of appropriate dimensions and satisfies that (C, A, B) and (F, A, D) are controllable and observable. For system (1), every admissible control $u(t)$ is given by a linear time-invariant system of the form

$$u = Js + Ky, \quad \dot{s} = Gs + Hy, \quad s(0) = 0 \tag{2}$$

which makes the closed-loop system given by (1) and (2) internally stable, where $s(t)$ is the state of the controller of a finite dimension; J, K, G and H are constant matrices of appropriate dimensions.

The control problem is to find an admissible control attenuating disturbances and initial state uncertainties in the way that, for given $N > 0$, $g = (z', u')'$ satisfies

$$\|g\|_2^2 < \|h\|_2^2 + x_0' N^{-1} x_0 \tag{3}$$

for all $h = (v', w')' \in L^2[0, \infty)$ and all $x_0 \in R^n$, s.t., $(v, w, x_0) \neq 0$. We call such an admissible control the disturbance and initial state uncertainty attenuation (DIA) control.

2.1 DIA Control

In order to solve the DIA control problem, we require the so-called Riccati equation conditions:

A1 : There exists a solution $M > 0$ to the Riccati equation

$$MA + A'M + F'F - M(MM' - DD')M = 0 \tag{4}$$

such that $A - BB'M + DD'M$ is stable.

A2 : There exists a solution $P > 0$ to the Riccati equation

$$PA' + AP + DD' - P(C'C - F'F)P = 0 \tag{5}$$

such that $A - PC'C + PF'F$ is stable.

A3 : $\rho(PM) < 1$,
where $\rho(X)$ denotes the spectral radius of matrix X, and $\rho(X) = \max|\lambda_i(X)|$.

In addition to these conditions, let us introduce the following condition:

A4 : $Q + N^{-1} - P^{-1} > 0$ where Q is the maximal solution of the Riccati equation

$$\begin{aligned} Q\left(A + DD'P^{-1}\right) + \left(A + DD'P^{-1}\right)' Q \\ -Q\left(DD' + LPC'CPL'\right)Q = 0 \end{aligned} \tag{6}$$

with $L := (I - PM)^{-1}$.

Theorem 1 *[3] Suppose that the conditions (A1), (A2), and (A3) are satisfied. The central control (7) is a DIA control if and only if the condition (A4) is satisfied, where the central control is given by*

$$\begin{aligned} u &= -B'S\underline{x} \\ \dot{\underline{x}} &= A\underline{x} + Bu + PC'(y - C\underline{x}) + PF'F\underline{x}, \\ & \quad \underline{x}(0) = 0, \quad S := M(I - PM)^{-1}. \end{aligned} \tag{7}$$

2.2 Parameterization of all DIA Controllers

Under the assumption that (A1)-(A3) are satisfied, the class of all H_∞ controls $u(t)$ are parametrized with a parameter Ψ as

$$u(t) = \underline{u}(t) + \left[\Psi\left(y - \underline{y}\right)\right](t) \tag{8}$$

$$\begin{aligned} \underline{u}(t) &= -B'S\underline{x}, \quad \underline{y}(t) = C(I + PS)\underline{x} \\ \dot{\underline{x}}(t) &= (A - BB'S - PC'C + PF'F)\underline{x} \\ & \quad + B\Psi\left(y - \underline{y}\right) + PC'y, \quad \underline{x}(0) = 0 \end{aligned} \tag{9}$$

Here, Ψ has a rational, strictly proper stable transfer function representation $\Psi(s)$, *s.t.* $||\Psi w||_2^2 < ||w||_2^2$, $\forall w \neq 0 \in L^2[0, \infty)$.

Theorem 2 *[3] Suppose that the conditions (A1)-(A3) are satisfied. An H_∞ control (8) with a parameter $\Psi(s)$ is a DIA control if and only if*

$$Q_{22} + N^{-1} - P^{-1} > 0 \tag{10}$$

where Q_{22} is the $(2,2)$ block of the maximal solution $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}$, whose existence is assured, of the Riccati equation

$$\begin{aligned} & Q\begin{bmatrix} A_m & 0 \\ -PSBK_m & A + DD'P^{-1} \end{bmatrix} \\ & + \begin{bmatrix} A_m & 0 \\ -PSBK_m & A + DD'P^{-1} \end{bmatrix}' Q \\ & -Q\begin{bmatrix} B_m B_m' & -B_m CPL' \\ -LPC'B_m' & DD' + LPC'CPL' \end{bmatrix} Q \\ & - \begin{bmatrix} K_m' K_m & 0 \\ 0 & 0 \end{bmatrix} = 0 \end{aligned} \tag{11}$$

for a minimal realization (A_m, B_m, K_m) of $\Psi(s)$, where A_m is stable, and $L = (I - PM)^{-1}$. Q_{22} is given independent of a particular choice of realization of $\Psi(s)$, and $Q_{22} \geq 0$.

3 System Description and Modeling

Magnetic suspension systems can suspend objects without any contact. Increasing use of this technology is now utilized for various industrial purposes, and has already applied to magnetically levitated vehicles, magnetic bearings, etc.

3.1 Construction

The experimental setup is shown in Fig.1[4]. An electromagnet is located at the top of the experimental system. The control problem is to levitate the iron ball stably utilizing the electromagnetic force, where a mass M of the iron ball is 1.75 kg, and steady state gap X is 5 mm. Note that this simple electromagnetic suspension system requires feedback control in order to be workable. As a gap sensor, a standard induction probe of eddy current type is placed below the ball.

3.2 Mathematical Model

In order to derive a model of the system by physical laws, we introduce following assumptions[4].

- **[a1]** Magnetic flux density and magnetic field do not have any hysteresis, and they are not saturated.
- **[a2]** There are no leakage flux in the magnetic circuit.

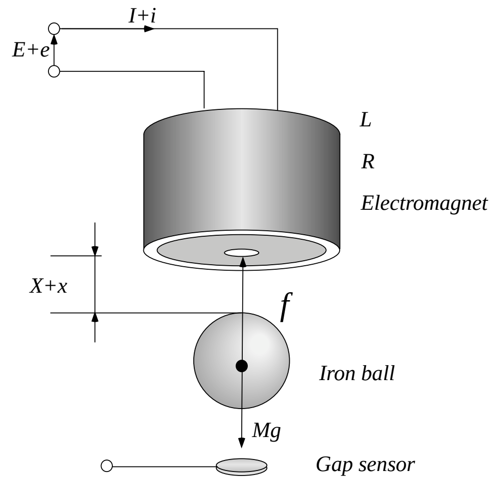

Figure 1: Magnetic Suspension System (M.S.S.)

- **[a3]** Magnetic permeability of the electromagnet is infinity.
- **[a4]** Eddy current in the magnetic pole can be neglected.
- **[a5]** Coil inductance is constant around the operating point, and an electromotive force due to a motion of the iron ball can be neglected.

These assumptions are almost essential to model this system. Under these assumptions, we derived equations of the motion, the electromagnetic force, and the electric circuit as

$$M\frac{d^2x}{dt^2} = Mg - f + v_m, \tag{12}$$

$$f = k\left(\frac{I+i}{X+x+x_0}\right)^2, \tag{13}$$

$$L\frac{di}{dt} + R(I+i) = E + e + v_L, \tag{14}$$

where M is a mass of the iron ball, X is a steady gap between the electromagnet(EM) and the iron ball, x is a deviation from X, I is a steady current, i is a deviation from I, E is a steady voltage, e is a deviation from E, f is EM force, k, x_0 are coefficients of f, L is an inductance of EM, and R is a resistance of EM, v_m and v_L are exogenous disturbance inputs.

Next we linearize the electromagnetic force (13) around the operating point by the Taylor series expansion as

$$f = k\left(\frac{I}{X+x_0}\right)^2 - K_x x + K_i i, \tag{15}$$

where $K_x = 2kI^2/(X+x_0)^3$ and $K_i = 2kI/(X+x_0)^2$. The sensor provides the information for the gap $x(t)$. Hence the measurement equation can be written as

$$y = x + w \tag{16}$$

where w represents the sensor noise as well as the model uncertainties. Thus, summing up the above results, the state equations for the system are

$$\begin{aligned} \dot{x}_g &= A_g x_g + B_g u_g + D_g v_0 \\ y_g &= C_g x_g + w \end{aligned} \tag{17}$$

where $x_g := [x \ \ \dot{x} \ \ i]'$, $u_g := e$, $v_0 := [v_m \ \ v_L]'$,

$$A_g = \begin{bmatrix} 0 & 1 & 0 \\ 4481 & 0 & -18.4 \\ 0 & 0 & -45.7 \end{bmatrix}, \quad B_g = \begin{bmatrix} 0 & 0 & 1.97 \end{bmatrix}'$$

$$C_g = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D_g = \begin{bmatrix} 0 & 0 \\ 0.57 & 0 \\ 0 & 1.97 \end{bmatrix}$$

Here $(A_g, \ B_g)$ and $(A_g, \ D_g)$ are controllable, and $(A_g, \ C_g)$ is observable.

4 Control System Design

4.1 Problem Setup

For the magnetic suspension system described and modeled in the previous section, our principal control objective is its stabilization. Further, as we have clarified in the modeling of the disturbances, it should be stabilized robustly against 1) unmodeled dynamics, 2) the neglected nonlinearities, 3) the parametric uncertainties. To this end, we will setup the control problem within the framework of the H_∞ DIA control.

First let us consider the system disturbance v_0. Since v_0 mainly acts on the plant in a low frequency range in practice, it is helpful to introduce a frequency weighting factor. Hence let v_0 be of the form

$$\begin{aligned} v_0 &= W_1(s)\, v(s) \\ W_1(s) &= \Phi W(s) = \Phi C_{w1}(sI - A_{w1})^{-1} B_{w1} \\ \Phi &= [1 \ \ 1]' \end{aligned} \tag{18}$$

where $W_1(s)$ is a frequency weighting whose gain is relatively large in a low frequency range. These values, as yet unspecified, can be regarded as free design parameters. Next we consider the variables which we want to regulate. In this study, since our main concern is in the stabilization of the iron ball, the gap and the corresponding velocity are chosen; i.e.,

$$z_g = F_g x_g, \quad F_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{19}$$

Then, as the error vector, let us define as follows

$$z = \Theta z_g, \quad \Theta = \mathrm{diag}\begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} \tag{20}$$

where Θ is a weighting matrix on the regulated variables z_g. This value, as yet unspecified, are also free design parameters. Finally, let $x := [x_g \ \ x_{w1}]'$, where

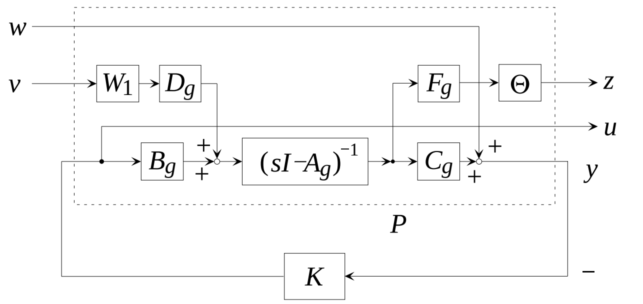

Figure 2: Generalized Plant

x_{w1} denotes the state of the frequency weighting $W_1(s)$, then we can construct the generalized plant as in the following;

$$\begin{aligned} \dot{x} &= Ax + Bu + Dv \\ y &= Cx + w \\ z &= Fx \end{aligned} \quad (21)$$

where

$$A = \begin{bmatrix} A_g & D_g C_{w1} \\ 0 & A_{w1} \end{bmatrix}, \; B = \begin{bmatrix} B_g \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} C_g & 0 \end{bmatrix}, \; D = \begin{bmatrix} D_g D_{w1} \\ B_{w1} \end{bmatrix}, \; F = \begin{bmatrix} \Theta F_g & 0 \end{bmatrix}$$

The block diagram of the generalized plant with an unspecified controller K is shown in Fig.2. Since the disturbances v and w represent the various model uncertainties, the effects of these disturbances on the error vector z should be reduced.

Now our control problem setup is: find an admissible controller $K(s)$ that attenuates disturbances and initial state uncertainties to achieve DIA condition in (3).

4.2 Design I: Central Controller

We design controllers for the generalized plant in the previous subsection based on the following 4-Step procedure.

[Step 1] Selection of the frequency weighting function $W(s)$: $W_1(s)$ is a frequency weighting whose gain is relatively large in a low frequency range.

[Step 2] Selection of the weighting Matrix Θ: Θ is a weighting matrix on the regulated variables z_g.

[Step 3] Construction of generalized plant: With the specified design parameters in Steps 1 and 2, the generalized plant is constructed. The DIA controller is designed for this plant.

[Step 4] Calculation of the maximum matrix N: Calculate the maximum N satisfies the condition (A4). For the sake of simplicity, the structure of the matrix N is limited in $N = nI$, where n is a positive scalar number.

4.2.1 DIA Controller 1: After some iteration in MATLAB environment, these parameters are chosen as follows;

$$W_1(s) = \frac{7.5}{s + 1.0e^{-4}}, \quad \Theta = \text{diag}\begin{bmatrix} 1.01 & 1.0e^{-5} \end{bmatrix} \quad (22)$$

Direct calculations yield the central controller;

$$K_{DIA_1} = C_{f1}(sI - A_{f1})^{-1} B_{f1} \quad (23)$$

where

$$\begin{aligned} A_{f1} &= A - BB'S - PC'C + PF'F \\ &= \begin{bmatrix} -1.38e^2 & 1.00 & 0 & 0 \\ -4.48e^3 & -2.98e^{-3} & -1.84e^1 & 4.28 \\ 1.05e^{11} & 1.98e^7 & -2.72e^4 & 6.33e^3 \\ -4.06e^{-2} & -2.71e^{-8} & 0 & -1e^{-4} \end{bmatrix} \\ B_{f1} &= PC' \\ &= \begin{bmatrix} -2.11e^9 & -1.41e^{11} & -2.07e^5 & -6.39e^5 \end{bmatrix}^T \\ C_{f1} &= -B'S \\ &= \begin{bmatrix} 1.68e^5 & 3.18e^1 & -4.36e^{-2} & 1.01e^{-2} \end{bmatrix} \end{aligned}$$

The frequency response of the controller K_{DIA_1} is shown in Fig. 3 by a solid line. And the maximum value of the weighting matrix N is $N = 3.855 \times 10^{-9} \times I$. We designed the standard H_∞ controller for the comparison, where the H_∞ controller[4] was designed via the MATLAB command `hinfsyn.m`. We denote the state-space realization of the obtained H_∞ controller as K_∞. The frequency response of the controller K_∞ is shown in Fig. 3 by a dotted line.

Comparing the controllers K_∞ and K_{DIA_1}, simulated step responses of these two controllers from the initial state $x_{02} = [x, \dot{x}, i]' = [0,\ 0,\ 0.1]'$ are shown in Fig.4, where the solid line shows a response with K_{DIA_1} and the dashed line shows one with K_∞. From this result, we can see that K_{DIA_1} achieves better performance against initial state uncertainty than K_∞ does.

4.2.2 Investigation of Weight N**:** The weighting matrix N on x_0 is a measure of relative importance of the initial-state uncertainty attenuation to the disturbance attenuation. A larger choice of N in the sense of matrix inequality order means finding an admissible control which attenuates the initial-state uncertainty more. For the evaluation of feedback performance against the weighting matrix N, we have designed another DIA controller K_{DIA_2}. After some iteration in MATLAB environment, design parameters are chosen as follows to obtain another DIA controller;

$$W_1(s) = \frac{6.75}{s + 1.0e^{-4}}, \quad \Theta = \text{diag}\begin{bmatrix} 1.025 & 1.0e^{-4} \end{bmatrix} \quad (24)$$

Direct calculations yield the central controller K_{DIA_2}, and its frequency response is shown in Fig. 3 by a dash-dot line.

The maximum value of Ns of the controllers K_{DIA_1} and K_{DIA_2} are given in Table 1.

The value of N for K_{DIA_1} is bigger than the value for K_{DIA_2}, which means that the controller K_{DIA_1} can attenuate the initial state uncertainty more than K_{DIA_2}. For the evaluation of this property, we examine the simulated time responses of the gap x with initial state uncertainty, where the initial state is $x_{02} = [x, \dot{x}, i]' = [0,\ 0,\ 0.1]'$ in Fig.5. Note that a scale of the vertical axis in Fig.5 is different from Fig.4.

The solid line shows the response with K_{DIA_1}, and the dashed line shows one with K_{DIA_2}. From this result, we can see that the controller K_{DIA_1}(solid line) which has a relatively large N, achieves greater attenuation of the initial state uncertainty, which means that the weighting matrix N can be an indicator of initial state uncertainty attenuation.

Table 1: DIA controllers and Ns

DIA controller	N
K_{DIA_1}	3.855×10^{-9}
K_{DIA_2}	2.677×10^{-9}

4.3 Design 2: Controller with a parameter Ψ

In this section, we design a DIA controller with a free parameter Ψ via experiment. The following step is added to the design procedure in Section 4.2.

[Step 5] Selection of the free parameter Ψ

After some iteration with experiments with a magnetic suspension system, these parameters are chosen as follows;

$$\begin{aligned} W_1(s) &= \frac{25.0}{s+10^{-2}}, \quad \Theta = \mathrm{diag}\left[\ 1.40 \quad 1.0e^{-3}\ \right] \\ \Psi(s) &= \frac{-9.9\times 10^{-2}}{s+0.1}. \end{aligned} \tag{25}$$

Here the free parameter Ψ, which satisfies (8), and has been chosen to let the controller have a integral property. Direct calculations yield the central controller $K_{DIA_3} = C_{f3}(sI - A_{f3})^{-1}B_{f3}$. Easy algebraic calculation with (8) and (9) derives the state space form of the DIA controller with a free parameter Ψ as

$$\begin{aligned} K_{DIA_3\Psi} &= C_{f3\Psi}(sI - A_{f3\Psi})^{-1}B_{f3\Psi} \qquad (26) \\ A_{f3\Psi} &= \begin{bmatrix} A - BB'S - PC'C + PF'F & BK_m \\ -B_mC(I+PS) & A_m \end{bmatrix} \\ B_{f3\Psi} &= \begin{bmatrix} PC' & B_m \end{bmatrix}, \quad C_{f3\Psi} = \begin{bmatrix} -B'S \\ K_m \end{bmatrix} \end{aligned}$$

The frequency response of the controller K_{DIA_3} and $K_{DIA_3\Psi}$ are shown in Fig.6, where a dashed line shows the frequency response of K_{DIA_3} and a solid line shows one of $K_{DIA_3\Psi}$. The controller gain of $K_{DIA_3\Psi}$ has been increased just as aimed, and is larger than K_{DIA_3} at the low frequency.

4.3.1 Experimental Evaluation: We have conducted experiments to evaluate properties of controllers K_{DIA_3} and $K_{DIA_3\Psi}$, using the experimental machine in Fig.1 The iron ball at a standstill has been suspended stably with either the controller K_{DIA_3} and $K_{DIA_3\Psi}$. To ascertain transient responses, we input the step reference signal to a suspended iron ball. It is expected that $K_{DIA_3\Psi}$ will show an improved response for the reference signal. A step reference signal is added to the system around 1[s], where the magnitude of the step signal is 0.5[mm], and steady state gap between the iron ball and the electromagnet is 5.0[mm]. Experimental results are shown in Figs.7 and 8, respectively. These figures show that both controllers give oscillatory responses and are not satisfactory. However both controllers maintain stable suspension, and our aim here is to investigate the difference of both K_{DIA_3} and $K_{DIA_3\Psi}$'s responses so as to evaluate the free parameter Ψ(s). In Fig.7, the steady state error has been left, but the Fig.8 shows that the controller $K_{DIA_3\Psi}$ makes this error to be zero, because of its integral property. This fact represents the usefulness of the free parameter Ψ.

5 Conclusion

In this paper, a robustness property of H_∞ controls against initial-state uncertainty was discussed. We evaluated the effectiveness of the proposed approach via a magnetic suspension system. First, the DIA controller has a relatively better transient property than the conventional standard H_∞ controller. Second, a role of the weighting matrix N for the initial state x_0 is shown via numerical simulation. N is a measure of relative importance of the initial-state uncertainty attenuation to the disturbance attenuation. A larger choice of N in the sense of matrix inequality order means finding an admissible control which attenuates the initial-state uncertainty more. Finally, usefulness and effectiveness of the free parameter Ψ of the mixed attenuation of disturbance and initial-state uncertainty has been examined via experimental results.

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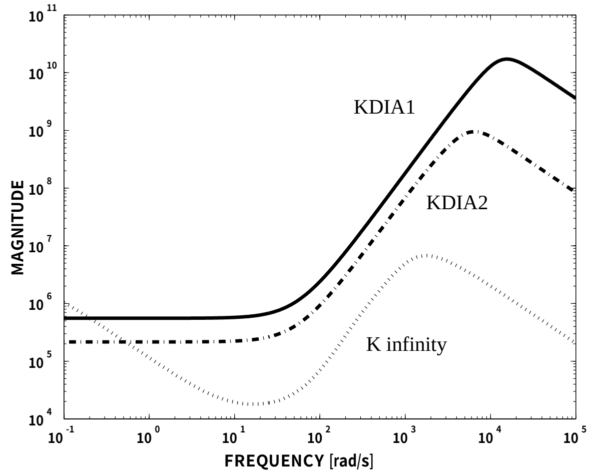

Figure 3: Frequency Response of the controller K_{DIA_1}, K_{DIA_2} and K_∞

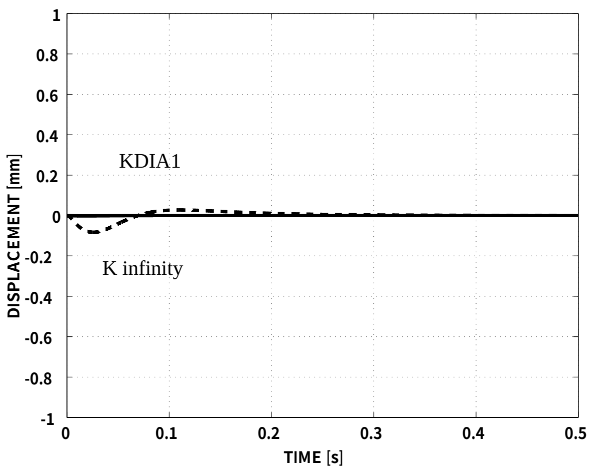

Figure 4: Comparison between K_{DIA_1} and K_∞

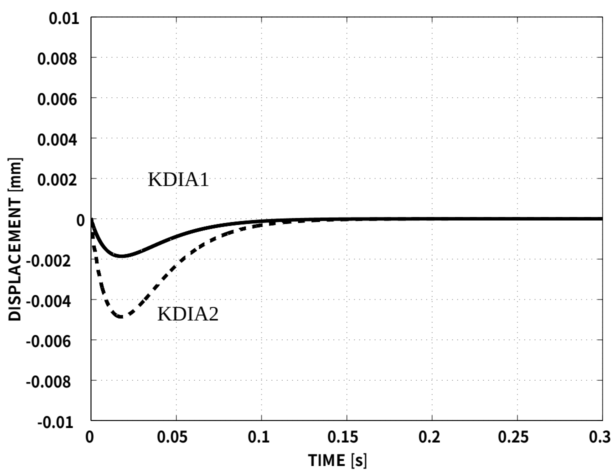

Figure 5: Initial Responses with K_{DIA_1}(solid line) and K_{DIA_2}(dashed line) for an initial state: $x_0 = [0.0,\ 0.0,\ 0.1]'$

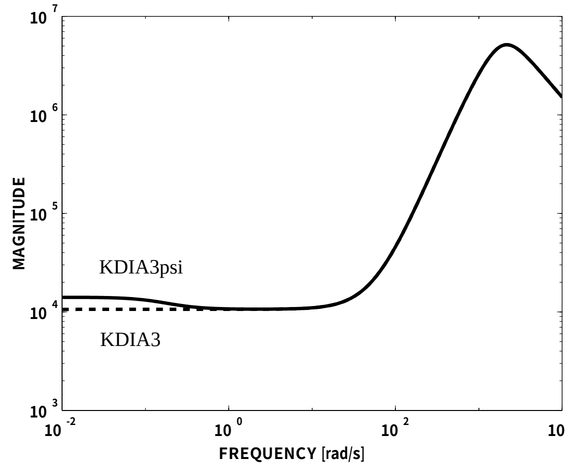

Figure 6: Frequency Responses of the controller K_{DIA_3}(Dashed line) and $K_{DIA_3\Psi}$(Solid line)

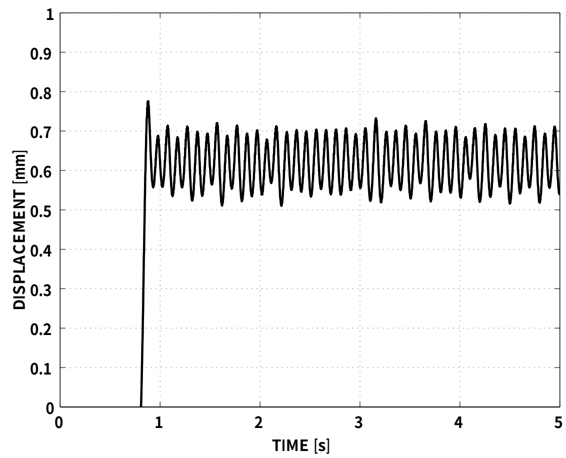

Figure 7: Experimental Results with $K_{DIA_3}(s)$ for Step Reference Signal(0.5[mm])

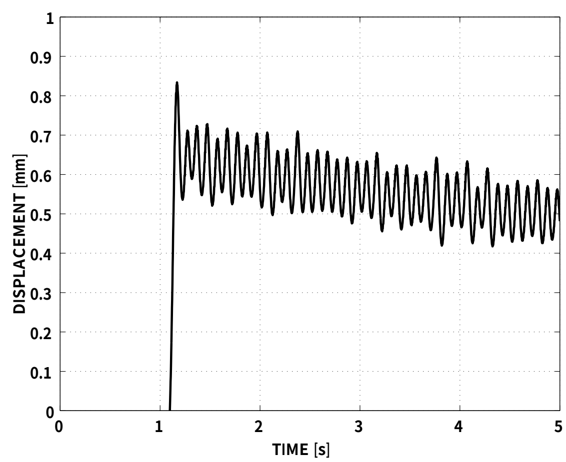

Figure 8: Experimental Results with $K_{DIA_3\Psi}(s)$ for Step Reference Signal(0.5[mm])