A Random Point Field related to Bose-Einstein Condensation

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Abstract

The random point fields which describe the position distributions of the systems of ideal boson gas in states of Bose-Einstein condensations are obtained through the thermodynamic limits. The resulting point fields are given by convolutions of two kinds of independent point fields: the so called boson processes whose generating functionals are represented by the inverses of the Fredholm determinants for operators related to the heat operator and the point fields whose generating functionals are represented by the resolvents of the operators. The construction of the latter point fields in an abstract formulation is also given.

Key words: random point field, classical statistical mechanics, continuum system, Boson process, Bose-Einstein condensation

1 Introduction

In the previous paper [7], which we will refer as I, the authors gave a method which derives typical kinds of random point fields, the boson and the fermion point processes on $\mathbb{R}^d$, through the thermodynamic limits of the random point fields of fixed finite numbers of points in bounded boxes in $\mathbb{R}^d$. The purpose

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of the paper is to give the random point fields which describe the position distributions of the systems of ideal boson gases in Bose-Einstein condensations [BEC] as an extension of I.

Let us consider the systems of N free bosons in a box of volume V in $\mathbb{R}^d$ and the quantum equilibrium states for the system of finite temperatures. Regarding the square of the absolute value of the wave function as the distribution function of the positions of N particles, we obtain random point fields of N points in the box. In I, we have shown that under the thermodynamic limit, $N, V \to \infty$ and $N/V \to \rho$, these random point fields converge weakly to the boson process μ_ρ^B whose Laplace transform is given by

$$\int_{Q(\mathbb{R}^d)} e^{-<f,\xi>} d\mu_\rho^B(\xi) = \mathrm{Det}[1 + \sqrt{1-e^{-f}} z_* G(1-z_*G)^{-1} \sqrt{1-e^{-f}}]^{-1} \quad (1.1)$$

if $\rho < \rho_c$, where $z_* \in (0,1)$ is determined by

$$\rho = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{z_* e^{-\beta|p|^2}}{1 - z_* e^{-\beta|p|^2}} = (z_* G(1-z_*G)^{-1})(x,x)$$

and ρ_c [see (3.6)] is the critical density above which the Bose-Einstein condensation takes place. The thermodynamic limits of the systems of fermions for all positive ρ have been also considered. As applications of the approach, the systems of para-particles and the systems of composite particles have been studied. The main apparatus is to apply the saddle point method to complex integrals related to the generalized Vere-Jones' formula [8,4]. The argument is based on the unified formulation of boson/fermion processes of [4]. For general references of this field, see e.g. [6] and references cited there in.

In this paper, we study the case of $\rho > \rho_c$ for the Boson systems in $\mathbb{R}^d, d > 2$, which corresponds to BEC. We need more technically elaborate analysis than in I about the largest eigenvalue $\tilde{g}_0(L)$ of the deformed heat operator $\tilde{G}_L$ in the box of size L. We must modify the saddle point method in I. The residue calculation is used instead of the gaussian integral. As the results of the thermodynamic limits, we get the random point fields on $\mathbb{R}^d$ which are given by the convolution of two kinds of independent point fields: 1. the boson processes whose generating functionals are represented by the inverses of the Fredholm determinants for operators related to the heat operator; 2. the point fields whose generating functionals are represented by the resolvents of the operators.

It would be interesting to consider profound relations between these two point fields. We have not succeeded in the analysis on the critical case $\rho = \rho_c$. These would be the subjects of future work.

The paper organized as follows: In §2 the construction of the point fields which appear in the resulting point fields as the second component (see above). The construction is made in a general framework of random point fields similar to that in [4], i.e., on the locally compact space of second countability. §3 is devoted to the analysis of the thermodynamic limit in $\mathbb{R}^d$.

2 Abstract formulation of the random point field

Let R be a locally compact Hausdorff space with countable basis and λ a positive Radon measure on R. We regard λ as a measure on the Borel σ-algebra $\mathcal{B}(R)$ which assigns finite values for compact sets. Relatively compact subsets of R will be called bounded. On $L^2(R;\lambda)$, we consider a (possibly unbounded) non-negative self-adjoint operator K which satisfies:

Condition K

(i) *[locally boundedness] For any bounded measurable subset Λ of R, the operator $K^{1/2}\chi_\Lambda$ is bounded, where χ_Λ denotes the operator multiplying the indicator function χ_Λ of Λ.*

(ii) $G = K(1+K)^{-1}$ *has a non-negative integral kernel $G(x,y)$ which satisfies*

$$\int_R G(x,y)\lambda(dy) \leqslant 1 \qquad \lambda - a.e.\, x \in R. \tag{2.1}$$

For a measurable function $f : R \to [0,\infty)$ with compact support and a bounded measurable set Λ satisfying $\Lambda \supset \operatorname{supp} f$, we have

$K^{1/2}\sqrt{1-e^{-f}} = K^{1/2}\chi_\Lambda\sqrt{1-e^{-f}}$ and hence that

$$K_f = \Big(K^{1/2}\sqrt{1-e^{-f}}\Big)^* K^{1/2}\sqrt{1-e^{-f}} \tag{2.2}$$

is a bounded non-negative self-adjoint operator. Here we regard $\sqrt{1-e^{-f}}$ as the multiplication operator of the function expressed by the same symbol. $Q(R)$ denotes the Polish space of all the locally finite non-negative integer valued Borel measures on R.

Theorem 2.1 *For R, λ and K which satisfy the above conditions and $\rho > 0$, there exists a unique Borel probability measure $\mu_{K,\rho}$ on $Q(R)$ such that*

$$\int_{Q(R)} e^{-<f,\xi>} d\mu_{K,\rho}(\xi) = \exp\Big(-\rho(\sqrt{1-e^{-f}}, [1+K_f]^{-1}\sqrt{1-e^{-f}})\Big) \tag{2.3}$$

holds for any non-negative measurable function f on R with compact support, where $(\,\cdot\,,\,\cdot\,)$ denotes the inner product of $L^2(R;\lambda)$.

Let us begin with some remarks before proving the theorem. It follows that G is self-adjoint and $0 \leqslant G \leqslant 1$, where 1 denotes the identity operator on $L^2(R;\lambda)$. Without loss of generality, we may assume that the $\mathcal{B}(R^2)$-measurable function $G(x,y)$ satisfies

$$\forall x, y \in R: \quad G(x,y) \geqslant 0, \quad G(x,y) = G(y,x)$$

and

$$\forall x \in R: \quad \int_R G(x,y)\,\lambda(dy) \leqslant 1.$$

Let us define the functions $G^n(x,y)$ inductively as $G^1(x,y) = G(x,y)$ and

$$G^{n+1}(x,y) = \int_R G^n(x,z)G(z,y)\,\lambda(dz) \quad \text{for} \quad n \in \mathbb{N}.$$

Then we have

$$\forall x, y \in R, \forall n \in \mathbb{N}: \quad G^n(x,y) \geqslant 0, \quad G^n(x,y) = G^n(y,x)$$

and

$$\forall x \in R, \forall n \in \mathbb{N}: \quad \int_R G^n(x,y)\,\lambda(dy) \leqslant 1.$$

It is obvious that $G^n(x,y)$ is the integral kernel of the operator G^n. Put

$$K_n = \sum_{k=1}^{n} G^k \quad \text{and} \quad K_n(x,y) = \sum_{k=1}^{n} G^k(x,y).$$

Then K_n is the bounded non-negative self-adjoint operator which has the non-negative integral kernel $K_n(x,y)$. The function

$$K(x,y) = \lim_{n\to\infty} K_n(x,y) = \sum_{k=1}^{\infty} G^k(x,y) \tag{2.4}$$

is well defined, if we admit infinity as its value.

Here we recall the following preliminary facts from functional analysis.

Lemma 2.2 (i) *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{L}(\mathcal{H})$ the Banach space of all the bounded operators on $\mathcal{H}$ and $\{A_n\}_{n\in\mathbb{N}}$ a bounded increasing sequence of non-negative self-adjoint operators in $\mathcal{L}(\mathcal{H})$. Then* s-$\lim_{n\to\infty} A_n$ *exists and is a bounded non-negative self-adjoint operator.*

(ii) *Suppose that $A_1, A_2, \cdots \in \mathcal{L}(L^2(R;\lambda))$ converge to $A \in \mathcal{L}(L^2(R;\lambda))$ strongly, A_n has the integral kernel $A_n(x,y)$ for each n and*

$$0 \leqslant A_n(x,y) \uparrow A(x,y) \qquad \lambda^{\otimes 2} - a.e.\,(x,y) \in R^2.$$

Then A has $A(x,y)$ as its integral kernel.

Proof : For (i), see e.g. [3]. For (ii), let $f \in L^2(R;\lambda)$. Then $|f| \in L^2(R;\lambda)$ and $(A_n|f|)(x) = \int A_n(x,y)|f(y)|\lambda(dy)$ holds. Taking the limit $n \to \infty$ (through a subsequence if necessary), we have $(A|f|)(x) = \int A(x,y)|f(y)|\,\lambda(dy)$ for λ-*a.e.* by strong convergence of the operators and the monotone convergence theorem. The a.e. boundedness of the integral in the right-hand side ensures the identity for f instead of $|f|$ by the dominated (instead of monotone) convergence theorem. □

Now we have the following proposition. Here and hereafter, $||\cdot||$ and $||\cdot||_T$ stand for the operator norm and the trace norm for operators, respectively, and $||\cdot||_p$ for the L^p-norm for functions.

Proposition 2.3 (i) *Put* $K_\Lambda = (K^{1/2}\chi_\Lambda)^* K^{1/2}\chi_\Lambda$ *for any bounded measurable* $\Lambda \subset R$. *Then,* K_Λ *is a bounded non-negative self-adjoint operator and has* $K_\Lambda(x,y) \equiv \chi_\Lambda(x)K(x,y)\chi_\Lambda(y)$ *as its integral kernel. The equality*

$$K_\Lambda = \sum_{k=1}^{\infty} \chi_\Lambda G^k \chi_\Lambda \tag{2.5}$$

holds in the sense of strong convergence of operators.

(ii) *For each* $k \in \mathbb{N}$, $H_k = \chi_\Lambda G((1-\chi_\Lambda)G)^{k-1}\chi_\Lambda$ *is a bounded non-negative self-adjoint operator having non-negative kernel, which is denoted by* $H_k(x,y)$. *The sum* $R_\Lambda = \sum_{k=1}^\infty H_k$ *exists in the strong convergence sense and is the bounded non-negative self-adjoint operator having non-negative kernel* $R_\Lambda(x,y) = \sum_{k=1}^\infty H_k(x,y)$.

(iii) $R_\Lambda = K_\Lambda(1+K_\Lambda)^{-1}, \quad ||R_\Lambda|| < 1.$

(iv) $(1+K_\Lambda)^{-1}\chi_\Lambda \geqslant 0$ *a.e. holds, where we regard* χ_Λ *as a function which belongs to* $L^2(R;\lambda)$.

Remark : From (i) of Proposition 2.3 and the argument above (2.2), it follows that $K_f = \sqrt{1-e^{-f}}K_\Lambda\sqrt{1-e^{-f}}$ and its kernel is given by

$\sqrt{1-e^{-f(x)}}K(x,y)\sqrt{1-e^{-f(y)}}$ for non-negative f satisfying $\operatorname{supp} f \subset \Lambda$.

Proof : (i) Boundedness and self-adjointness of K_Λ are obvious.

Using the spectral decomposition $K = \int_0^\infty \lambda\, dE_\lambda$, we have

$$G = \int_0^\infty \frac{\lambda}{1+\lambda}\, dE_\lambda.$$

Hence,

$$||\sum_{k=1}^{n} \chi_\Lambda G^k \chi_\Lambda|| = \sup_{||\phi||_2=1} \sum_{k=1}^{n} (\chi_\Lambda\phi, G^k\chi_\Lambda\phi)$$

$$= \sup_{||\phi||_2=1} \int \sum_{k=1}^{n} \left(\frac{\lambda}{1+\lambda}\right)^k d(\chi_\Lambda\phi, E_\lambda\chi_\Lambda\phi) \leqslant \sup_{||\phi||_2=1} \int \lambda\, d(\chi_\Lambda\phi, E_\lambda\chi_\Lambda\phi)$$

$$= \sup_{||\phi||_2=1} ||K^{1/2}\chi_\Lambda\phi||_2^2 = ||K_\Lambda||.$$

Since $\chi_\Lambda G^k\chi_\Lambda \geqslant 0$ holds for every $k \in \mathbb{N}$, Lemma 2.2(i) yields the existence of s-$\lim_{n\to\infty}\sum_{k=1}^n \chi_\Lambda G^k\chi_\Lambda$. On the other hand, thanks to the monotone convergence theorem, we get (2.5) in the weak sense:

$$(\phi, \sum_{k=1}^{n} \chi_\Lambda G^k\chi_\Lambda\phi) = \int \sum_{k=1}^{n} \left(\frac{\lambda}{1+\lambda}\right)^k d(\chi_\Lambda\phi, E_\lambda\chi_\Lambda\phi)$$

$$\longrightarrow \int \lambda\, d(\chi_\Lambda\phi, E_\lambda\chi_\Lambda\phi) = (\phi, K_\Lambda\phi).$$

Thus we have (2.5) in the strong sense.

Lemma 2.2(ii) yields the assertion on the kernel of K_Λ.

(ii) It is obvious that H_k is a bounded non-negative self-adjoint operator for every $k \in \mathbb{N}$. From the non-negativity of the kernel of G^k, we have the non-negativity of the kernel $H_k(x,y)$ and

$$0 \leqslant H_k(x,y) \leqslant \chi_\Lambda(x)G^k(x,y)\chi_\Lambda(y).$$

From Lemma 2.2(i) and the estimate

$$||\sum_{k=1}^{n} H_k|| = \sup_{||\phi||_2=1} \sum_{k=1}^{n} \int_{R^2} \overline{\phi(x)} H_k(x,y)\phi(y)\lambda^{\otimes 2}(dx\,dy)$$

$$\leqslant \sup_{||\phi||_2=1} \sum_{k=1}^{n} \int_{R^2} |\phi(x)|\chi_\Lambda(x)G^k(x,y)\chi_\Lambda(y)|\phi(y)|\lambda^{\otimes 2}(dx\,dy)$$

$$\leqslant ||\sum_{k=1}^{n} \chi_\Lambda G^k\chi_\Lambda|| \leqslant ||K_\Lambda||,$$

we get the existence of the strong limit R_Λ of $\{\sum_{k=1}^n H_k\}_n$ and its bounded self-adjointness.

Lemma 2.2(ii) yields the assertion on the kernel of R_Λ.

(iii) From

$$\sum_{k=1}^{n} H_k - \sum_{k=1}^{n} \chi_\Lambda G^k\chi_\Lambda = \sum_{k=1}^{n} \chi_\Lambda G[((1-\chi_\Lambda)G)^{k-1} - G^{k-1}]\chi_\Lambda$$

$$= \sum_{k=2}^{n}\sum_{l=1}^{k-1} \chi_\Lambda G((1-\chi_\Lambda)G)^{k-l-1}(-\chi_\Lambda G)G^{l-1}\chi_\Lambda$$

$$= -\sum_{l=1}^{n-1}\sum_{k=l+1}^{n} \chi_\Lambda G((1-\chi_\Lambda)G)^{k-l-1}\chi_\Lambda\chi_\Lambda G^l\chi_\Lambda = -\sum_{l=1}^{n-1}\sum_{m=1}^{n-l} H_m\chi_\Lambda G^l\chi_\Lambda,$$

we get the relation

$$\sum_{k=1}^{n} H_k(x,y) - \sum_{k=1}^{n}\chi_\Lambda(x)G^k(x,y)\chi_\Lambda(y)$$

$$= -\sum_{l=1}^{n-1}\sum_{m=1}^{n-l}\int_R H_m(x,z)\chi_\Lambda(z)G^l(z,y)\chi_\Lambda(y)\lambda(dz) \quad a.e.$$

among the kernels. Taking the limit $n \to \infty$, we have

$$R_\Lambda(x,y) - K_\Lambda(x,y) = -\int_R R_\Lambda(x,z)K_\Lambda(z,y)\lambda(dz) \qquad \lambda^{\otimes 2} - a.e.(x,y)$$

by the monotone convergence theorem. It implies $R_\Lambda = K_\Lambda(1+K_\Lambda)^{-1}$. Since K_Λ is non-negative and bounded, $||R_\Lambda|| < 1$.

(iv) We may regard G as a contraction operator on $L^\infty(R;\lambda)$ because of (2.1). H_k is also contraction on $L^\infty(R;\lambda)$ for all $k \in \mathbb{N}$. Thus we have

$$\sum_{k=1}^{n}(H_k\chi_\Lambda)(x) \leqslant \sum_{k=1}^{n}(H_k\chi_\Lambda)(x) + \Big(\chi_\Lambda G((1-\chi_\Lambda)G)^{n-1}(1-\chi_\Lambda)\Big)(x)$$

$$\leqslant \sum_{k=1}^{n-1}(H_k\chi_\Lambda)(x) + \Big(\chi_\Lambda G((1-\chi_\Lambda)G)^{n-2}(1-\chi_\Lambda)\Big)(x)$$

$$\leqslant \cdots \leqslant (\chi_\Lambda G1)(x) \leqslant \chi_\Lambda(x),$$

where the non-negativity of the kernel of G and (2.1) have been used. On the other hand, we get $\sum_{k=1}^n H_k\chi_\Lambda \to R_\Lambda\chi_\Lambda \quad a.e.$ from (ii) through subsequence if necessary. Hence $(1+K_\Lambda)^{-1}\chi_\Lambda = \chi_\Lambda - R_\Lambda\chi_\Lambda \geqslant 0 \quad a.e.$ holds. □

(*Proof of Theorem 2.1*) Recall that $K_f = \sqrt{1-e^{-f}}K_\Lambda\sqrt{1-e^{-f}}$, for non-negative measurable f and a bounded measurable set $\Lambda \supset \operatorname{supp} f$. Since

$$(1+(1-e^{-f})K_\Lambda)\sqrt{1-e^{-f}}(1+K_f)^{-1}\sqrt{1-e^{-f}}$$

$$= 1-e^{-f} = 1-e^{-f}R_\Lambda - e^{-f}(1+K_\Lambda)^{-1}$$

and

$$1+(1-e^{-f})K_\Lambda = (1-e^{-f}R_\Lambda)(1+K_\Lambda),$$

we get

$$
\begin{aligned}
&\sqrt{1-e^{-f}}(1+K_f)^{-1}\sqrt{1-e^{-f}} \\
&=(1+K_\Lambda)^{-1}(1-e^{-f}R_\Lambda)^{-1}(1-e^{-f}R_\Lambda-e^{-f}(1+K_\Lambda)^{-1}) \\
&=(1+K_\Lambda)^{-1}[1-(1-e^{-f}R_\Lambda)^{-1}e^{-f}(1+K_\Lambda)^{-1}] \\
&=(1+K_\Lambda)^{-1}-(1+K_\Lambda)^{-1}\sum_{n=0}^{\infty}(e^{-f}R_\Lambda)^n e^{-f}(1+K_\Lambda)^{-1}.
\end{aligned}
$$

The Neumann expansion in the last step is valid since $||e^{-f}R_\Lambda|| \leqslant ||R_\Lambda|| < 1$. Hence we have

$$
\begin{aligned}
&-(\sqrt{1-e^{-f}},[1+K_f]^{-1}\sqrt{1-e^{-f}}) \\
&=-(\chi_\Lambda,(1+K_\Lambda)^{-1}\chi_\Lambda)+\sum_{l=0}^{\infty}((1+K_\Lambda)^{-1}\chi_\Lambda,e^{-f}(R_\Lambda e^{-f})^l(1+K_\Lambda)^{-1}\chi_\Lambda).
\end{aligned}
$$

Substituting this identity to the right-hand side of (2.3), expanding the exponential and symmetrizing, we get an expression of the form

$$
\sum_{n=0}^{\infty}\frac{1}{n!}\int_{\Lambda^n}\sigma_{\Lambda^n}(x_1,\cdots,x_n)e^{-\sum_{k=1}^{n}f(x_k)}dx_1\cdots dx_n \tag{2.6}
$$

with a family of symmetric non-negative functions $\{\sigma_{\Lambda^n}\}$ for every $\Lambda \supset \operatorname{supp} f$. For the existence of the measure $\mu_{K,\rho}$ on $Q(R)$, it is enough to show the consistency condition[2]:

$$
\sigma_{\Lambda^n}(x_1,\cdots,x_n)=\sum_{l=0}^{\infty}\frac{1}{l!}\int_{\Delta^l}\sigma_{(\Lambda\cup\Delta)^{n+l}}(x_1,\cdots,x_n,y_1,\cdots,y_l)\,dy_1\cdots dy_l,
$$

where $\Delta\cap\Lambda=\emptyset$. This condition can be derived easily from the facts that the right hand side of (2.3) does not depend on $\Lambda \supset \operatorname{supp} f$ and that for a given Λ, $\{\sigma_{\Lambda^n}\}$ in (2.6) is uniquely determined a.e., since f can be arbitrary non-negative measurable function satisfying $\operatorname{supp} f \subset \Lambda$.

Thus we have proved Theorem 2.1. □

3 The Thermodynamic Limit

In this section, we follow the arguments and the notation of I §2.2. However, let us review them briefly to make the article self-contained.

Consider $\mathcal{H}_L = L^2(\Lambda_L)$ on $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$ for $d > 2$ with the Lebesgue measure on Λ_L. Let $\triangle_L$ be the Laplacian under the periodic boundary condition in $\mathcal{H}_L$. For $k \in \mathbb{Z}^d$, $\varphi_k^{(L)}(x) = L^{-d/2}\exp(i2\pi k\cdot x/L)$ is an eigenfunction of $\triangle_L$, and $\{\,\varphi_k^{(L)}\,\}_{k\in\mathbb{Z}^d}$ forms a complete orthonormal system [CONS] of $\mathcal{H}_L$. The operator $G_L = \exp(\beta\triangle_L)$ has the integral kernel

$$G_L(x,y) = \sum_{k\in\mathbb{Z}^d} e^{-\beta|2\pi k/L|^2}\varphi_k^{(L)}(x)\overline{\varphi_k^{(L)}(y)}, \tag{3.1}$$

for $\beta > 0$. We put $g_k^{(L)} = \exp(-\beta|2\pi k/L|^2)$ which is the eigenvalue of G_L for the eigenfunction $\varphi_k^{(L)}(x)$. We also need the operator $G = \exp(\beta\triangle)$ on $L^2(\mathbb{R}^d)$ and its integral kernel

$$G(x,y) = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} e^{-\beta|p|^2+ip\cdot(x-y)} = \frac{\exp(-|x-y|^2/4\beta)}{(4\pi\beta)^{d/2}}.$$

Here we consider only periodic boundary conditions, though we can deal with Dirichlet or Neumann boundary conditions for rectangles in the same way.

Let $f : \mathbb{R}^d \to [0,\infty)$ be a continuous function of compact support. We will only consider the case where L is so large that Λ_L contains $\operatorname{supp} f$. We regard f as a function on Λ_L naturally. Let

$$\tilde{G}_L = G_L^{1/2} e^{-f} G_L^{1/2}, \tag{3.2}$$

where e^{-f} represents the operator of multiplication by the function e^{-f}.

It is obvious from the non-negativity of f that $0 \leqslant \tilde{G}_L \leqslant G_L$. Let us denote all the eigenvalues of $\tilde{G}_L$ in decreasing order

$$\tilde{g}_0(L) \geqslant \tilde{g}_1(L) \geqslant \cdots \geqslant \tilde{g}_j(L) \geqslant \cdots .$$

Correspondingly, we relabel the eigenvalues $\{\, g_k^{(L)} \,\}_{k\in\mathbb{Z}^d}$ of G_L as

$$g_0(L) = 1 > g_1(L) \geqslant \cdots \geqslant g_j(L) \geqslant \cdots .$$

By the min-max principle, we have

$$g_j(L) \geqslant \tilde{g}_j(L) \qquad (j = 0, 1, 2, \cdots).$$

Note that $\varphi_0^{(L)}$ has the eigenvalue $g_0(L) = g_0^{(L)} = 1$.

Suppose there are N identical particles which obey Bose-Einstein statistics in Λ_L with periodic boundary conditions at the inverse temperature β. The basic postulates of quantum mechanics and statistical mechanics of canonical ensembles yield

$$p_{L,N}^B(x_1,\cdots,x_N) = \frac{1}{Z_B N!}\mathrm{per}\,\{G(x_i,x_j)\}_{1\leqslant i,j\leqslant N} \tag{3.3}$$

as the probability density distribution of the positions of N particles of the system, where Z_B is the normalization constant and per represents the permanent of matrices. Here, we have set $\hbar^2/2m = 1$. We define the random point field (the probability measure on $Q(\mathbb{R}^d)$) $\mu^B_{L,N}$ induced by the map $\Lambda_L^N \ni (x_1, \cdots, x_N) \mapsto \sum_{j=1}^N \delta_{x_j} \in Q(\mathbb{R}^d)$ from the probability measure on Λ_L^N which has the density (3.3). By $\mathrm{E}^B_{L,N}$, we denote the expectation with respect to $\mu^B_{L,N}$. The Laplace transform of the point process is given by

$$\begin{aligned}
&\mathrm{E}^B_{L,N}\Big[e^{-<f,\xi>}\Big] \\
&= \frac{\int_{\Lambda_L^N} \exp(-\sum_{j=1}^N f(x_j)) \mathrm{per}\,\{G_L(x_i, x_j)\}_{i,j=1}^N \, dx_1 \cdots dx_N}{\int_{\Lambda_L^N} \mathrm{per}\,\{G_L(x_i, x_j)\}_{i,j=1}^N \, dx_1 \cdots dx_N} \\
&= \frac{\int_{\Lambda_L^N} \mathrm{per}\,\{\tilde{G}_L(x_i, x_j)\}_{i,j=1}^N \, dx_1 \cdots dx_N}{\int_{\Lambda_L^N} \mathrm{per}\,\{G_L(x_i, x_j)\}_{i,j=1}^N \, dx_1 \cdots dx_N}.
\end{aligned} \tag{3.4}$$

Let us consider the thermodynamic limit, where N and the volume of the box Λ_L tend to infinity in such a way that the densities tend to a positive finite value ρ:

$$L,\ N \to \infty, \quad N/L^d \to \rho > 0. \tag{3.5}$$

In this paper, we concentrate on the high density region

$$\rho > \rho_c = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{-\beta|p|^2}}{1 - e^{-\beta|p|^2}} \tag{3.6}$$

where the Bose-Einstein condensation takes place.

Theorem 3.1 (i) *The operator $K = G(1-G)^{-1}$ is a non-negative unbounded self-adjoint operator in $L^2(\mathbb{R}^d)$ and satisfies the Condition K in §2. Moreover, K_f defined by (2.2) is a trace class operator.*

(ii) *The finite point fields defined above converge weakly to the random point field whose Laplace transform is given by*

$$\mathrm{E}^B_\rho\Big[e^{-<f,\xi>}\Big] = \frac{\exp\Big(-(\rho - \rho_c)(\sqrt{1-e^{-f}}, [1+K_f]^{-1}\sqrt{1-e^{-f}})\Big)}{\mathrm{Det}[1+K_f]}$$

in the thermodynamic limit (3.5–3.6).

Remark 1: Thus the resulting point field of the theorem is a convolution of a point field which is an example of those discussed in §2 and a boson process. On the formulation of boson processes, we refer to [4], where the operator K is assumed to be bounded, however the proof given there is also valid for the present case.

Remark 2: It might be plausible to think whether there are any profound reason between these two random point fields. However, we have no idea about it. We have not succeeded in the analysis on the critical case $\rho = \rho_c$. These would be the subjects of future work.

We begin the proof with the following lemma, where we use the notation

$$\Box_k^{(L)} \;=\; \frac{2\pi}{L}\Big(k + \Big(-\frac{1}{2}, \frac{1}{2}\Big]^d\Big) \qquad \text{for} \quad k \in \mathbb{Z}^d.$$

Lemma 3.2 *For $z \in [0,1]$, $\nu = 1, 2$ and $L \in [1, \infty)$, let us define functions $a_\nu(\,\cdot\,; z), a_\nu^{(L)}(\,\cdot\,; z)$ on $\mathbb{R}^d$ by*

$$a_\nu(p; z) = \frac{z e^{-\beta|p|^2}}{(1 - z e^{-\beta|p|^2})^\nu}$$

and

$$a_\nu^{(L)}(p; z) = \begin{cases} 0 & \text{if} \quad p \in \Box_0^{(L)} \\ a_\nu(2\pi k/L; z) & \text{if} \quad p \in \Box_k^{(L)} \quad \text{for} \quad k \in \mathbb{Z}^d - \{0\}. \end{cases}$$

Then

$$0 \leqslant a_1^{(L)}(p; z) \leqslant a_1(2p/(2+\sqrt{d}); 1) \in L^1(\mathbb{R}^d)$$

and the bounds for large L

$$\frac{L^d}{(2\pi)^d}\int_{\mathbb{R}^d} a_2^{(L)}(p; z)\, dp \leqslant \ell(L) \equiv \begin{cases} c_d(L/\sqrt{\beta})^d & \text{if} \quad d > 4 \\ \tilde{c}_4(L/\sqrt{\beta})^4 \log(\tilde{c}L/\sqrt{\beta}) & \text{if} \quad d = 4 \\ c_d(L/\sqrt{\beta})^4 & \text{if} \quad d < 4 \end{cases}$$

hold, where $c_d, \tilde{c}_4$ and $\tilde{c}$ are positive constants.

Proof: Since a_ν is monotone increasing in z and monotone decreasing as a function of $|p|$, we have

$$a_\nu^{(L)}(p; z) \leqslant \sup_{L \geqslant 1} a_\nu^{(L)}(p; 1) \leqslant \sup\{\, a_\nu(q; 1) \,|\, q \in \mathbb{R}^d, L \geqslant 1,$$

$$|q| \geqslant 2\pi/L, |q - p| \leqslant (2\pi/L)(\sqrt{d}/2)\,\}$$

$$\leqslant \sup\{\, a_\nu(q; 1) \,|\, q \in \mathbb{R}^d, L \geqslant 1, |q| \geqslant 2\pi/L,\ |p| - \pi\sqrt{d}/L \leqslant |q|\,\}.$$

In the case of $|p| \geqslant (2+\sqrt{d})\pi$, the last supremum is attained at $L = 1, |q| = |p| - \pi\sqrt{d}$ then $|q| \geqslant |2p|/(2+\sqrt{d})$ holds. On the other hand, if $|p| < (2+\sqrt{d})\pi$, the supremum is attained at $L = (2+\sqrt{d})\pi/|p|$, $|q| = 2\pi/L$ and then $|q| = |2p|/(2+\sqrt{d})$ holds. For both cases, we get the bound $a_\nu^{(L)}(p; z) \leqslant a_\nu(2p/(2+\sqrt{d}); 1)$. Since $d > 2$, we get $a_1(2p/(2+\sqrt{d}); 1) \in L^1(\mathbb{R}^d)$.

Integrating the angular variables, we have

$$\frac{L^d}{(2\pi)^d}\int_{\mathbb{R}^d} a_2^{(L)}(p; z)\, dp \leqslant \frac{L^d}{(2\pi)^d}\int_{|p| \geqslant \pi/L} a_2(2p/(2+\sqrt{d}); 1) dp$$

$$= \left(\frac{L}{2\pi\sqrt{\beta'}}\right)^d S_d \int_{\pi\sqrt{\beta'}/L}^{\infty} \frac{q^{d-1}e^{-q^2}}{(1-e^{-q^2})^2} dq = \left(\frac{L}{2\pi\sqrt{\beta'}}\right)^d S_d \mathcal{I}_d,$$

where $\beta' = 4\beta/(2+\sqrt{d})^2$. Since $\mathcal{I}_d \leqslant \int_0^\infty [q^{d-1}e^{-q^2}/(1-e^{-q^2})^2]\, dq < \infty$ for $d > 4$; $\mathcal{I}_d \leqslant \int_{\pi\sqrt{\beta'}/L}^\infty [q^{d-1}/q^4]\, dq = (4-d)^{-1}(L/\pi\sqrt{\beta'})^{4-d}$ for $d < 4$ and

$$\mathcal{I}_4 \leqslant \int_1^\infty \frac{q^{3-1}e^{-q^2}}{(1-e^{-q^2})^2} dq + \int_{\pi\sqrt{\beta'}/L}^1 \frac{q^3}{q^4} dq = const. + \log\frac{L}{\pi\sqrt{\beta'}}, \tag{3.7}$$

we get the bounds for $\pi\sqrt{\beta} \leqslant L$. □

In the following, $||\cdot||_T$ stands for the trace norm.

(*Proof of Theorem 3.1(i)*) It is obvious that $K = G(1-G)^{-1}$ is a unbounded non-negative self-adjoint operator satisfying $G = K(1+K)^{-1}$. In fact, K is explicitly given by the Fourier transformation:

$$K\phi = \mathcal{F}^{-1}(a_1(\,\cdot\,;1)\mathcal{F}\phi)$$

for

$$\phi \in \mathrm{Dom}\, K = \{\, \psi \in L^2(\mathbb{R}^d) \,|\, a_1(\,\cdot\,;1)\mathcal{F}\psi \in L^2(\mathbb{R}^d)\,\}.$$

Condition K(ii) for G is also obvious.

Let us show the locally boundedness of K. For bounded measurable $\Lambda \subset \mathbb{R}^d$,

$$||\sqrt{K}\chi_\Lambda\phi||_2^2 = ||\sqrt{a_1(\,\cdot\,;1)}\mathcal{F}(\chi_\Lambda\phi)||_2^2 \leqslant ||\sqrt{a_1(\,\cdot\,;1)}||_2^2||\mathcal{F}(\chi_\Lambda\phi)||_\infty^2$$

$$\leqslant ||a_1(\,\cdot\,;1)||_1||\chi_\Lambda\phi||_1^2 \leqslant (2\pi)^d\rho_c||\chi_\Lambda||_2^2||\phi||_2^2.$$

Thus $K^{1/2}\chi_\Lambda$ is bounded. $K(x,y)$ in (2.4) is given by

$$K(x,y) = \sum_{n=1}^\infty G^n(x,y) = \sum_{n=1}^\infty \int \frac{dp}{(2\pi)^d} e^{-n\beta|p|^2 + ip\cdot(x-y)}$$

$$= \int \frac{dp}{(2\pi)^d} a_1(p;1)e^{ip\cdot(x-y)},$$

where we have used the dominated convergence theorem. From $a_1 \in L^1(\mathbb{R}^d)$, $K(x,y)$ is continuous. The remark after Proposition 2.3 and the continuity of f yield that the kernel of K_f is continuous. Hence K_f is a trace class operator, because $||K_f||_T = \int K_f(x,x)\, dx = \rho_c||1-e^{-f}||_1 < \infty$. □

The rest of this section is devoted to the proof of the second part of the theorem. Put

$$D_L = G_L - \tilde{G}_L = G_L^{1/2}(1-e^{-f})G_L^{1/2}, \quad W_L = G_L^{1/2}\sqrt{1-e^{-f}},$$

then $D_L = W_L W_L^*$. Note also that

$$\frac{L^d}{(2\pi)^d}\int_{\mathbb{R}^d} a_\nu^{(L)}(p;z)\,dp = \sum_{k\in\mathbb{Z}^d-\{0\}} \frac{zg_k^{(L)}}{(1-zg_k^{(L)})^\nu} \tag{3.8}$$

$$= ||zQ_0G_LQ_0(1-zQ_0G_LQ_0)^{-\nu}||_T.$$

Here P_0 is the orthogonal projection on $L^2(\Lambda_L)$ to the one dimensional subspace $\mathbb{C}\varphi_0^{(L)}$ and $Q_0 = 1 - P_0$.

Lemma 3.3 *It holds that the bound*

(i) $||Q_0G_LQ_0(1-Q_0G_LQ_0)^{-2}||_T \leqslant \ell(L)$

and the following convergences in the limit $L\to\infty$:

(ii) $$L^{-d}\mathrm{Tr}G_L \longrightarrow \int_{\mathbb{R}^d} e^{-\beta|p|^2}dp/(2\pi)^d = \sqrt{4\pi\beta}^{-d},$$

$$||D_L||_T \longrightarrow ||1-e^{-f}||_1/\sqrt{4\pi\beta}^d.$$

(iii) $$L^{-d}||Q_0G_LQ_0(1-Q_0G_LQ_0)^{-1}||_T \longrightarrow \rho_c < \infty.$$

(iv) *If $\{z_L\}\subset(0,1)$ and $z_L\to 1$, then*

$$\sup_{x,y\in\Lambda} |[z_LQ_0G_LQ_0(1-z_LQ_0G_LQ_0)^{-1}](x,y) - K(x,y)| \to 0$$

for any fixed bounded measurable set $\Lambda\subset\mathbb{R}^d$.

Proof : (i)–(iii) are immediate consequences of the above remarks, Lemma 3.2 and the dominated convergence theorem.

For (iv), put $e(p;x) = e^{ip\cdot x}$ and

$$e^{(L)}(p;x) = e(2\pi k/L;x) \quad \text{if} \quad p\in\square_k^{(L)} \quad \text{for} \quad k\in\mathbb{Z}^d.$$

Then Lemma 3.2 and the dominated convergence theorem also yield

$$\begin{aligned}
&|[z_LQ_0G_LQ_0(1-z_LQ_0G_LQ_0)^{-1}](x,y) - K(x,y)|\\
\leqslant &\int \frac{dp}{(2\pi)^d}|e^{(L)}(p;x-y)a_1^{(L)}(p;z_L) - e(p;x-y)a_1(p;1)|\\
\leqslant &\int \frac{dp}{(2\pi)^d}\Big(|a_1^{(L)}(p;z_L) - a_1(p;1)| + |e^{(L)}(p;x-y) - e(p;x-y)|a_1(p;1)\Big)\\
&\longrightarrow 0.
\end{aligned}$$

□

In the followings, we use the notation $B_L = \hat{O}(L^\alpha)$ which means

$$\exists c_1 \geqslant c_2 > 0 : c_1 L^\alpha \geqslant B_L \geqslant c_2 L^\alpha.$$

Lemma 3.4

(i) *For large L,* $\quad g_0(L) - \tilde{g}_0(L) = L^{-d}(\sqrt{1-e^{-f}},$
$[1 + W_L^* Q_0[1 - Q_0 G_L Q_0]^{-1} Q_0 W_L]^{-1}\sqrt{1-e^{-f}})\,(1+o(1))$
$= (\varphi_0^{(L)}, (D_L - D_L Q_0[1 - Q_0 \tilde{G}_L Q_0]^{-1} Q_0 D_L)\varphi_0^{(L)})\,(1+o(1)).$

(ii) $\quad ||1-e^{-f}||_1/L^d = (\varphi_0^{(L)}, D_L\varphi_0^{(L)})$
$\geqslant (\varphi_0^{(L)}, D_L Q_0[\tilde{g}_0(L) - Q_0\tilde{G}_L Q_0]^{-1} Q_0 D_L \varphi_0^{(L)})$

(iii) $L^d(g_0(L) - \tilde{g}_0(L)) \in \left[\dfrac{||1-e^{-f}||_1(1+o(1))}{1+\rho_c||1-e^{-f}||_1}, ||1-e^{-f}||_1\right]$

(iv) *Let $\tilde{\varphi}_0^{(L)}$ be the normalized eigenfunction of $\tilde{G}_L$ for eigenvalue $\tilde{g}_0(L)$ such that $(\tilde{\varphi}_0^{(L)}, \varphi_0^{(L)}) \geqslant 0$. Put $\tilde{\varphi}_0^{(L)} = a\varphi_0^{(L)} + \varphi'$,*
$\varphi_0^{(L)} = a'\tilde{\varphi}_0^{(L)} + \tilde{\varphi}' \quad (\,(\varphi_0^{(L)}, \varphi') = 0, \quad (\tilde{\varphi}_0^{(L)}, \tilde{\varphi}') = 0\,).$
Then $a = a'$ and $||\varphi'||^2 = ||\tilde{\varphi}'||^2 = 1 - a^2 = O(L^{-2d}\ell(L))$ hold.

Proof: Here, we suppress the index L in $g_j(L), \tilde{g}_j(L), \varphi_0^{(L)}$ and so on. First notice that $(\varphi_0, D_L\varphi_0) = ||1-e^{-f}||_1/L^d$. From the min-max principle, $d > 2$ and the value of $g_1 = \exp(-\beta|2\pi/L|^2)$, we have

$$g_0 = 1 \geqslant \tilde{g}_0 \geqslant (\varphi_0, \tilde{G}_L\varphi_0) = 1 - (\varphi_0, D_L\varphi_0) = 1 - \hat{O}(L^{-d}) \qquad (3.9)$$

$$> g_1 = 1 - \hat{O}(L^{-2}) \geqslant \tilde{g}_1$$

for L large enough. Hence the eigenspace of $\tilde{G}_L$ for the largest eigenvalue $\tilde{g}_0$ is one-dimensional. Let $\tilde{\varphi}_0$ be the normalized eigenfunction for $\tilde{g}_0$ and put $\tilde{\varphi}_0 = a\varphi_0 + \varphi' \quad (\,(\varphi_0, \varphi') = 0\,)$. Then $\tilde{G}_L\tilde{\varphi}_0 = \tilde{g}_0\tilde{\varphi}_0$ yields

$$a\tilde{G}_L\varphi_0 + \tilde{G}_L\varphi' = a\tilde{g}_0\varphi_0 + \tilde{g}_0\varphi'.$$

Applying P_0 and Q_0, we have

$$\begin{aligned} ag_0 - a(\varphi_0, D_L\varphi_0) - (\varphi_0, D_L\varphi') &= a\,\tilde{g}_0 \\ -aQ_0 D_L\varphi_0 + Q_0\tilde{G}_L\varphi' &= \tilde{g}_0\,\varphi'. \end{aligned}$$

Because $Q_0\tilde{G}_L Q_0 \leqslant Q_0 G_L Q_0 \leqslant g_1 < \tilde{g}_0$ and $\tilde{g}_0 - Q_0\tilde{G}_L Q_0$ is positive invertible,

$$\varphi' = -a[\tilde{g}_0 - Q_0\tilde{G}_L Q_0]^{-1}Q_0 D_L\varphi_0, \tag{3.10}$$

$$\begin{aligned} g_0 - \tilde{g}_0 &= (\varphi_0, (D_L - D_L Q_0[\tilde{g}_0 - Q_0\tilde{G}_L Q_0]^{-1}Q_0 D_L)\varphi_0) \\ &= (W_L^*\varphi_0, (1 - W_L^* Q_0[\tilde{g}_0 - Q_0\tilde{G}_L Q_0]^{-1}Q_0 W_L)W_L^*\varphi_0). \end{aligned} \tag{3.11}$$

For brevity, we put

$$X' = W_L^* Q_0[\tilde{g}_0 - Q_0 G_L Q_0]^{-1}Q_0 W_L, \quad X = W_L^* Q_0[1 - Q_0 G_L Q_0]^{-1}Q_0 W_L$$

and

$$\tilde{X} = W_L^* Q_0[\tilde{g}_0 - Q_0\tilde{G}_L Q_0]^{-1}Q_0 W_L.$$

Then we have

$$\tilde{X} - X' = -\tilde{X}X',$$

and hence

$$\tilde{X} = X'(1 + X')^{-1} \quad \text{and} \quad 1 - \tilde{X} = (1 + X')^{-1}. \tag{3.12}$$

Together with $W_L^*\varphi_0 = \sqrt{1 - e^{-f}}L^{-d/2}$, we have

$$g_0 - \tilde{g}_0 = L^{-d}(\sqrt{1 - e^{-f}}, (1 + X')^{-1}\sqrt{1 - e^{-f}}) \tag{3.13}$$

from (3.11).

Now, we want to replace X' by X in the right hand side. From (3.9), $1 - \tilde{g}_0 = O(L^{-d})$ and $\tilde{g}_0 - g_1 = \hat{O}(L^{-2})$ hold. Note also that we have $\sum_{k\neq 0} g_k/(1 - g_k)^2 \leqslant \ell(L)$ from Lemma 3.3(i). It follows that

$$||X' - X|| = (1 - \tilde{g}_0)||W_L^* Q_0[\tilde{g}_0 - Q_0 G_L Q_0]^{-1}[1 - Q_0 G_L Q_0]^{-1}Q_0 W_L||$$

$$= (1 - \tilde{g}_0) \sup_{||\phi||_2 = 1} (\phi, W_L^* Q_0[\tilde{g}_0 - Q_0 G_L Q_0]^{-1}[1 - Q_0 G_L Q_0]^{-1}Q_0 W_L\phi)$$

$$\begin{aligned} &\leqslant (1 - \tilde{g}_0) \sup_{||\phi||_2 = 1} \sum_{k\neq 0} |(\varphi_k, \sqrt{1 - e^{-f}}\phi)|^2 \frac{g_k}{(\tilde{g}_0 - g_k)(1 - g_k)} \\ &\leqslant (1 - \tilde{g}_0) \sup_{||\phi||_2 = 1} \frac{||\sqrt{1 - e^{-f}}\phi||_1^2}{L^d} \frac{1 - g_1}{\tilde{g}_0 - g_1} \sum_{k\neq 0} \frac{g_k}{(1 - g_k)^2} \\ &= ||1 - e^{-f}||_1 O(L^{-2d}\ell(L)) = o(1). \end{aligned} \tag{3.14}$$

Together with the similar estimate $||X|| \leqslant \rho_c||1 - e^{-f}||_1(1 + o(1))$, we have $||X'|| \leqslant \rho_c||1 - e^{-f}||_1(1 + o(1))$. Thus (3.13) yields

$$L^d(g_0 - \tilde{g}_0) = (\sqrt{1 - e^{-f}}, (1 + X')^{-1}\sqrt{1 - e^{-f}}) \geqslant \frac{||1 - e^{-f}||_1}{1 + \rho_c||1 - e^{-f}||_1}(1 + o(1)),$$

which is the lower bound of (iii). The upper bound of (iii) is obvious.

From

$$|(\sqrt{1-e^{-f}},(1+X')^{-1}\sqrt{1-e^{-f}})-(\sqrt{1-e^{-f}},(1+X)^{-1}\sqrt{1-e^{-f}})|$$

$$\leqslant ||\sqrt{1-e^{-f}}||_2^2||(1+X)^{-1}||\,||(1+X')^{-1}||\,||X-X'||=o(1),$$

we get the first equality of (i). Replacing X' by X in (3.13) and tracing the argument back to (3.11), we get the second one of (i).

The bound (ii) is an immediate consequence of $g_0 \geqslant \tilde{g}_0$ and (3.11).

(iv) Clearly, $a=(\tilde{\varphi}_0,\varphi_0)=a'$. As for (3.12), we have

$$(\tilde{g}_0-Q_0\tilde{G}_LQ_0)^{-1}Q_0W_L=(\tilde{g}_0-Q_0G_LQ_0)^{-1}Q_0W_L(1+X')^{-1}. \tag{3.15}$$

This and estimates similar to (3.14) derive the bound

$$||\varphi'||^2=a^2(\varphi_0,D_LQ_0[\tilde{g}_0-Q_0\tilde{G}_LQ_0]^{-2}Q_0D_L\varphi_0)$$

$$\begin{aligned}
&\leqslant a^2||W_L^*\varphi_0||_2^2||W_L^*Q_0[\tilde{g}_0-Q_0\tilde{G}_LQ_0]^{-2}Q_0W_L||\\
&=a^2||W_L^*\varphi_0||_2^2||(1+X')^{-1}W_L^*Q_0[\tilde{g}_0-Q_0G_LQ_0]^{-2}Q_0W_L(1+X')^{-1}||\\
&=a^2O(L^{-2d}\ell(L))
\end{aligned}$$

from (3.10). Now the bound for $1-a^2$ is obvious. □

As in I, we use the generalized Vere-Jones' formula [8,4] in the form

$$\frac{1}{N!}\int \mathrm{per}\,(J(x_i,x_j))_{i,j=1}^N\lambda^{\otimes N}(dx_1\cdots dx_N)=\oint_{S_r(0)}\frac{dz}{2\pi i z^{N+1}}\mathrm{Det}(1-zJ)^{-1},$$

where $r>0$ satisfies $||rJ||<1$. $S_r(\zeta)$ denotes the integration contour defined by the map $\theta\mapsto\zeta+r\exp(i\theta)$, where θ ranges from $-\pi$ to π, $r>0$ and $\zeta\in\mathbb{C}$. Then we get

$$\mathrm{E}_{L,N}^B\Big[e^{-<f,\xi>}\Big]=\frac{z_0^N}{\tilde{z}_0^N}\frac{\mathrm{Det}[1-z_0G_L]}{\mathrm{Det}[1-\tilde{z}_0\tilde{G}_L]}$$

$$\times\frac{\oint_{S_1(0)}\mathrm{Det}\Big[1-\tilde{z}_0\tilde{G}_L(1-\tilde{z}_0\tilde{G}_L)^{-1}(\eta-1)\Big]^{-1}d\eta/2\pi i\eta^{N+1}}{\oint_{S_1(0)}\mathrm{Det}\Big[1-z_0G_L(1-z_0G_L)^{-1}(\eta-1)\Big]^{-1}d\eta/2\pi i\eta^{N+1}}. \tag{3.16}$$

The positive real numbers $z_0=z_0(L,N)$ and $\tilde{z}_0=\tilde{z}_0(L,N)$ are chosen as the solutions of the equations

$$\mathrm{Tr}_{\mathcal{H}}\Big[z_0G_L(1-z_0G_L)^{-1}\Big]=\mathrm{Tr}_{\mathcal{H}}\Big[\tilde{z}_0\tilde{G}_L(1-\tilde{z}_0\tilde{G}_L)^{-1}\Big]=N. \tag{3.17}$$

In fact, the following lemma holds. Hereafter, we will often suppress the (L,N)-dependence in $z_0(L,N)$ and $\tilde{z}_0(L,N)$ for brevity.

Lemma 3.5 (i) *The parameter* $z_0 \in (0,1)$ *is uniquely determined by the equation*

$$\mathrm{Tr}\Big[z_0 G_L (1 - z_0 G_L)^{-1}\Big] = N. \tag{3.18}$$

(ii) *The parameter* $\tilde{z}_0 \in (0, \tilde{g}_0^{-1}(L)\,)$ *is uniquely determined by the equation*

$$\mathrm{Tr}\Big[\tilde{z}_0 \tilde{G}_L (1 - \tilde{z}_0 \tilde{G}_L)^{-1}\Big] = N. \tag{3.19}$$

(iii) $0 \leqslant \tilde{z}_0 - z_0 = O(L^{-d})$

(iv) $1 - z_0 = (1 + o(1))L^{-d}(\rho - \rho_c)^{-1}$

Proof : Let $H(z_0)$ and $\tilde{H}(\tilde{z}_0)$ be the left-hand sides of (3.18) and (3.19) , respectively. Since H is a monotone increasing continuous function on $[0,1)$, $H(0) = 0$ and $H(1-0) = \infty$, (i) follows. (ii) is similar.

The first inequality of (iii) is a consequence of $H(z) \geqslant \tilde{H}(z)$ for $z \in [0,1)$.

To show the second part of (iii) and (iv), let us make the following remark on the thermodynamic limit (3.5).

(a) *If and only if* $\rho < \rho_c$, $\{z_0(L,N)\}$ *converges to* $z = z_* \in (0,1)$*, the unique solution of*

$$\rho = \int \frac{dp}{(2\pi)^d} a_1(p;z)$$

in the thermodynamic limit (3.5)

(b) *If and only if* $\rho > \rho_c$, $L^d(1 - z_0) \longrightarrow 1/(\rho - \rho_c)$. In this case, $\lim z_0 = 1$ holds.

(c) *If and only if* $\rho = \rho_c$, $\lim z_0 = 1$ *and* $L^d(1 - z_0) \longrightarrow +\infty$.

To show (a – c), note that

$$\frac{z_0}{L^d(1 - z_0)} + \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} a_1^{(L)}(p;z_0) = \mathrm{Tr}[z_0 G_L (1 - z_0 G_L)^{-1}]/L^d \to \rho. \tag{3.20}$$

We have that

$$\int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} a_1^{(L)}(p;z_0) \to \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} a_1(p;z)$$

for $\lim z_0 = z \in [0,1]$ by the dominated convergence theorem, and that the limit is a strictly increasing function of z. (See Lemma 3.2.) If $\lim z_0 = z_* \in [0,1)$, the limit of (3.20) tends to

$$\rho = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} a_1(p;z_*) < \rho_c.$$

If $\lim z_0 = 1$, then $\rho = \rho_c + \lim z_0/L^d(1-z_0) \geqslant \rho_c$. Now suppose $\{z_0(L,N)\}$ does not converge. Then by taking converging subsequences having different limits, we deduce a contradiction to (3.20). Thus we get the classification (a – c) and (iv).

Now we have the second part of (iii) using Lemma 3.4(iii),

$$z_0 = 1 - \hat{O}(L^{-d}) \leqslant \tilde{z}_0 < \tilde{g}_0^{-1} = 1 + \hat{O}(L^{-d}). \qquad \square$$

In order to understand the subsequent arguments, it is helpful to keep the followings in mind:

$$g_0 = 1, \quad g_1 = 1 - \hat{O}(L^{-2}) \geqslant Q_0 G_L Q_0 \geqslant Q_0 \tilde{G}_L Q_0 \qquad \text{(see (3.9))}$$

$$\tilde{g}_0 = 1 - \hat{O}(L^{-d}) \qquad \text{(Lemma 3.4(iii))}$$

$$z_0 = 1 - \hat{O}(L^{-d}) \quad \tilde{z}_0 = z_0 + O(L^{-d}) \qquad \text{(Lemma 3.5(iii, iv))}$$

$$(\varphi_k^{(L)}, D_L \varphi_k^{(L)}) = g_k^{(L)} ||1 - e^{-f}||_1 / L^d$$

Lemma 3.6

$$\text{(i) } P_0[1 - \tilde{z}_0 \tilde{G}_L]^{-1} P_0 = \Big(\frac{1}{1 - \tilde{z}_0 \tilde{g}_0} + O(L^{-d}\ell(L)) \Big) P_0,$$

$$\text{(ii) } ||Q_0[1 - \tilde{z}_0 \tilde{G}_L]^{-1}|| = ||[1 - \tilde{z}_0 \tilde{G}_L]^{-1} Q_0|| = O(\sqrt{\ell(L)}),$$

$$||Q_0[1 - z_0 \tilde{G}_L]^{-1}|| = ||[1 - z_0 \tilde{G}_L]^{-1} Q_0|| = O(\sqrt{\ell(L)}),$$

$$\text{(iii) } \mathrm{Tr}(Q_0[1 - z_0 G_L]^{-1} D_L [1 - z_0 G_L]^{-1} Q_0) = O(L^{-d}\ell(L)).$$

Proof: (i) By lemma 3.4(iv), we have

$$|(\varphi_0, (1 - \tilde{z}_0 \tilde{G}_L)^{-1} \varphi_0) - (1 - \tilde{z}_0 \tilde{g}_0)^{-1}|$$

$$= |(a\tilde{\varphi}_0 + \tilde{\varphi}', (1 - \tilde{z}_0 \tilde{G}_L)^{-1}(a\tilde{\varphi}_0 + \tilde{\varphi}')) - (1 - \tilde{z}_0 \tilde{g}_0)^{-1}|$$

$$\leqslant \frac{1 - a^2}{1 - \tilde{z}_0 \tilde{g}_0} + |(\tilde{\varphi}', (1 - \tilde{z}_0 \tilde{G}_L)^{-1} \tilde{\varphi}')|$$

$$\leqslant ((1 - \tilde{z}_0 \tilde{g}_0)^{-1} + (1 - \tilde{z}_0 \tilde{g}_1)^{-1}) O(L^{-2d}\ell(L)) = O(L^{-d}\ell(L)),$$

where we have used

$$\frac{1}{1 - \tilde{z}_0 \tilde{g}_0} + \frac{1}{1 - \tilde{z}_0 \tilde{g}_1} \leqslant 2 + \mathrm{Tr}[\tilde{z}_0 \tilde{G}_L (1 - \tilde{z}_0 \tilde{G}_L)^{-1}] = 2 + N = O(L^d),$$

in the last step.

(ii) Note that $Q_0\tilde{\varphi}_0 = \varphi'$ in the notation of lemma 3.4(iv). Then we get

$$||Q_0(1-\tilde{z}_0\tilde{G}_L)^{-1}|| \leqslant \frac{||\varphi'||}{1-\tilde{z}_0\tilde{g}_0} + \frac{1}{1-\tilde{z}_0\tilde{g}_1} = O(L^d\sqrt{L^{-2d}\ell(L)}) + O(L^2).$$

The second bound is obtained similarly.

(iii) From the equality just above Lemma 3.6, the left-hand side equals

$$\frac{||1-e^{-f}||_1}{L^d}\sum_{k\neq 0}\frac{g_k}{(1-z_0 g_k)^2},$$

which yields the right-hand side by Lemma 3.3(i). $\square$

We need a finer estimate than Lemma 3.5(iii).

Lemma 3.7 *The asymptotic behaviors*

$$\text{(i) } \tilde{z}_0 - z_0 = (1-\tilde{g}_0)(1+o(1)),$$
$$\text{(ii) } 1-\tilde{z}_0 g_0' = (1-\tilde{z}_0\tilde{g}_0)(1+o(1)) = (1-z_0)(1+o(1)) = \frac{1+o(1)}{L^d(\rho-\rho_c)}$$

hold, where

$$g_0' = 1-(\varphi_0^{(L)}, D_L\varphi_0^{(L)}) + \tilde{z}_0(\varphi_0^{(L)}, D_L Q_0[1-\tilde{z}_0 Q_0\tilde{G}_L Q_0]^{-1}Q_0 D_L\varphi_0^{(L)}).$$

Proof: (i) Let us begin with

$$0 = N - N = \mathrm{Tr}[\tilde{z}_0\tilde{G}_L(1-\tilde{z}_0\tilde{G}_L)^{-1} - z_0 G_L(1-z_0 G_L)^{-1}] =$$
$$(\varphi_0, ((1-\tilde{z}_0\tilde{G}_L)^{-1} - (1-z_0 G_L)^{-1})\varphi_0) + \mathrm{Tr}[Q_0((1-\tilde{z}_0\tilde{G}_L)^{-1} - (1-z_0\tilde{G}_L)^{-1})Q_0]$$
$$+\mathrm{Tr}[Q_0((1-z_0\tilde{G}_L)^{-1} - (1-z_0 G_L)^{-1})Q_0].$$

The first term of the right hand side equals

$$(1-\tilde{z}_0\tilde{g}_0)^{-1} - (1-z_0 g_0)^{-1} + O(L^{-d}\ell(L))$$
$$= \frac{(\tilde{z}_0 - z_0)\tilde{g}_0 - z_0(g_0-\tilde{g}_0)}{(1-\tilde{z}_0\tilde{g}_0)(1-z_0 g_0)} + O(L^{-d}\ell(L))$$

by Lemma 3.6(i). On the other hand, the second term has the bound

$$(\tilde{z}_0 - z_0)|\mathrm{Tr}[Q_0(1-\tilde{z}_0\tilde{G}_L)^{-1}\tilde{G}_L(1-z_0\tilde{G}_L)^{-1}Q_0]|$$
$$\leqslant \frac{\tilde{z}_0 - z_0}{\tilde{z}_0}||\tilde{z}_0\tilde{G}_L(1-\tilde{z}_0\tilde{G}_L)^{-1}||_T||(1-z_0\tilde{G}_L)^{-1}Q_0||$$

$$= O(L^{-d}L^d\sqrt{\ell(L)}) = o(L^d)$$

by Lemma 3.5(iii, ii) and Lemma 3.6(ii). The third term can be estimated as

$$|\mathrm{Tr}[Q_0((1 - z_0\tilde{G}_L)^{-1} - (1 - z_0G_L)^{-1})Q_0]|$$

$$= z_0|\mathrm{Tr}[Q_0(1 - z_0\tilde{G}_L)^{-1}W_LW_L^*(1 - z_0G_L)^{-1}Q_0]|$$

$$= z_0|\mathrm{Tr}[Q_0(1 - z_0G_L)^{-1}W_L(1 + z_0W_L^*(1 - z_0G_L)^{-1}W_L)^{-1}W_L^*(1 - z_0G_L)^{-1}Q_0]|$$

$$\leqslant z_0||Q_0(1 - z_0G_L)^{-1}W_LW_L^*(1 - z_0G_L)^{-1}Q_0]||_T = O(L^{-d}\ell(L)) = o(L^d),$$

where we have used a equality similar to (3.15) and Lemma 3.6(iii). Thus we have

$$\frac{z_0(g_0 - \tilde{g}_0) - (\tilde{z}_0 - z_0)\tilde{g}_0}{(1 - \tilde{z}_0\tilde{g}_0)(1 - z_0g_0)} = o(L^d).$$

On the other hand, $(1 - \tilde{z}_0\tilde{g}_0)(1 - z_0g_0) = O(L^{-2d})$ holds. Thus we have

$$z_0(g_0 - \tilde{g}_0) - (\tilde{z}_0 - z_0)\tilde{g}_0 = o(L^{-d}).$$

Note that $g_0 - \tilde{g}_0$ is exactly of order L^{-d} by Lemma 3.4(iii), we get the desired estimate.

(ii) From (3.11), we have

$$|\tilde{g}_0 - g_0'| = |(\varphi_0, D_LQ_0[(\tilde{g}_0 - Q_0\tilde{G}_LQ_0)^{-1} - (\tilde{z}_0^{-1} - Q_0\tilde{G}_LQ_0)^{-1}]Q_0D_L\varphi_0)|$$

$$= |(\varphi_0, D_LQ_0(\tilde{g}_0 - Q_0\tilde{G}_LQ_0)^{-1/2}[(\tilde{z}_0^{-1} - \tilde{g}_0)(\tilde{z}_0^{-1} - Q_0\tilde{G}_LQ_0)^{-1}]$$
$$\times(\tilde{g}_0 - Q_0\tilde{G}_LQ_0)^{-1/2}Q_0D_L\varphi_0)|$$

$$\leqslant |\tilde{z}_0^{-1} - \tilde{g}_0|||(\tilde{z}_0^{-1} - Q_0\tilde{G}_LQ_0)^{-1}||(\varphi_0, D_LQ_0(\tilde{g}_0 - Q_0\tilde{G}_LQ_0)^{-1}Q_0D_L\varphi_0)$$

$$\leqslant O(L^{-d})O(L^2)(\varphi_0, D_L\varphi_0) = O(L^{2-2d}) = o(L^{-d}),$$

where Lemma 3.4(ii) has been used in the last inequality. Hence, we obtain $1 - \tilde{z}_0g_0' = 1 - \tilde{z}_0\tilde{g}_0 + o(L^{-d})$. On the other hand, we have

$$1 - \tilde{z}_0\tilde{g}_0 = 1 - z_0 + [\tilde{z}_0(1 - \tilde{g}_0) - (\tilde{z}_0 - z_0)] = \frac{1 + o(1)}{L^d(\rho - \rho_c)} + o(L^{-d}),$$

thanks to Lemma 3.5(iv) and (i) above. □

Put $p_j^{(N)} = z_0g_j(L)/(1 - z_0g_j(L))$, $\tilde{p}_j^{(N)} = \tilde{z}_0\tilde{g}_j(L)/(1 - \tilde{z}_0\tilde{g}_j(L))$, then by Lemma 3.5(i, ii) 3.7(ii), we have $\sum_{j=0}^{\infty} p_j^{(N)} = \sum_{j=0}^{\infty} \tilde{p}_j^{(N)} = N$,

$$p_0^{(N)} = \hat{O}(L^d), \quad \tilde{p}_0^{(N)} = \hat{O}(L^d),\ p_0^{(N)}/\tilde{p}_0^{(N)} = 1 + o(1) \tag{3.21}$$

and $p_1^{(N)} = \hat{O}(L^2) \geqslant p_2^{(N)} \geqslant \cdots$, $\tilde{p}_1^{(N)} = O(L^2) \geqslant \tilde{p}_2^{(N)} \geqslant \cdots$.

Lemma 3.8 *In this notation, it holds that*

$$\oint_{S_1(0)} \frac{1}{\mathrm{Det}\Big[1 - z_0 G_L (1 - z_0 G_L)^{-1}(\eta - 1)\Big]} \frac{d\eta}{2\pi i \eta^{N+1}} = \frac{1 + o(1)}{e p_0^{(N)}},$$
$$\oint_{S_1(0)} \frac{1}{\mathrm{Det}\Big[1 - \tilde{z}_0 \tilde{G}_L (1 - \tilde{z}_0 \tilde{G}_L)^{-1}(\eta - 1)\Big]} \frac{d\eta}{2\pi i \eta^{N+1}} = \frac{1 + o(1)}{e \tilde{p}_0^{(N)}}.$$

Proof : Set $R^{(N)} = \tilde{R}^{(N)} = L^{(d-2)/2}$. Since $\sum_{j=1}^{\infty} p_j^{(N)}(1 + p_j^{(N)})$ $= \mathrm{Tr}[z_0 Q_0 G_L Q_0 (1 - z_0 Q_0 G_L Q_0)^{-2}] \leqslant \sum_{j=1}^{\infty} g_j/(1 - g_j)^2$, we get

$$\frac{R^{(N)2} \sum_{j=1}^{\infty} p_j^{(N)}(1 + p_j^{(N)})}{p_0^{(N)2}} \to 0$$

by $p_0^{(N)} = \hat{O}(L^d)$ and Lemma 3.3(i). Then Lemma A.2 yields

$$\text{the l.h.s. of the 1st eq.} = \oint_{S_1(0)} \frac{1}{\prod_{j=0}^{\infty}(1 - p_j^{(N)}(\eta - 1))} \frac{d\eta}{2\pi i \eta^{N+1}} = \frac{1 + o(1)}{e p_0^{(N)}}.$$

For the second equality, we notice that $\tilde{p}_j^{(N)} \leqslant (1 + o(1)) p_j^{(N)}$ holds for all $j = 1, 2, \cdots$, because of $z_0, \tilde{z}_0 = 1 + O(L^{-d})$ and $\tilde{g}_j^{(N)} \leqslant g_j^{(N)} \leqslant 1 - \hat{O}(L^{-2})$. Together with (3.21), we have

$$\frac{\tilde{R}^{(N)2} \sum_{j=1}^{\infty} \tilde{p}_j^{(N)}(1 + \tilde{p}_j^{(N)})}{\tilde{p}_0^{(N)2}} \leqslant (1 + o(1)) \frac{R^{(N)2} \sum_{j=1}^{\infty} p_j^{(N)}(1 + p_j^{(N)})}{p_0^{(N)2}} \to 0.$$

Thus the second equality also follows from Lemma A.2. $\square$

Now we have

$$\mathrm{E}_{L,N}^{B}\Big[e^{-<f,\xi>}\Big] = \frac{z_0^N}{\tilde{z}_0^N} \frac{\mathrm{Det}[1 - z_0 G_L]}{\mathrm{Det}[1 - \tilde{z}_0 \tilde{G}_L]} (1 + o(1))$$

from (3.16), (3.21) and the above lemma. Since P_0, Q_0 and G_L commute, $\mathrm{Det}[1 - z_0 G_L] = (1 - z_0)\mathrm{Det}[1 - z_0 Q_0 G_L Q_0]$. We use the Feshbach formula to get

$$\mathrm{Det}[1 - \tilde{z}_0 \tilde{G}_L] = \mathrm{Det} \begin{pmatrix} P_0 - \tilde{z}_0 P_0 \tilde{G}_L P_0 & -\tilde{z}_0 P_0 \tilde{G}_L Q_0 \\ -\tilde{z}_0 Q_0 \tilde{G}_L P_0 & Q_0 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0 \end{pmatrix}$$
$$= \mathrm{Det}_{Q_0 \mathcal{H}_L}[Q_0 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]$$
$$\times \mathrm{Det}_{P_0 \mathcal{H}_L}[P_0 - \tilde{z}_0 P_0 \tilde{G}_L P_0 - \tilde{z}_0 P_0 \tilde{G}_L Q_0 (Q_0 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0)^{-1} \tilde{z}_0 Q_0 \tilde{G}_L P_0]$$

$$= \mathrm{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]$$
$$\times\Big(1 - \tilde{z}_0[1 - (\varphi_0^{(L)}, D_L\varphi_0^{(L)}) + \tilde{z}_0(\varphi_0^{(L)}, D_L Q_0[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]^{-1} Q_0 D_L \varphi_0^{(L)})]\Big)$$

where Det is the Fredholm determinant for operators on $\mathcal{H}_L$ and $\mathrm{Det}_{Q_0\mathcal{H}_L}$ for operators on the subspace $Q_0\mathcal{H}_L$, etc. Now from Lemma 3.7(ii) and Lemma 3.5(iii, iv), we get

$$\mathrm{E}_{L,N}^{B}\Big[e^{-<f,\xi>}\Big] = \frac{z_0^N}{\tilde{z}_0^N}\frac{(1-z_0)\mathrm{Det}[1 - z_0 Q_0 G_L Q_0]}{(1-\tilde{z}_0 g_0')\mathrm{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]}(1+o(1))$$

$$= \frac{z_0^N}{\tilde{z}_0^N}\frac{\mathrm{Det}[1 - z_0 Q_0 G_L Q_0]}{\mathrm{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]}(1+o(1))$$

$$= \exp\Big(-\frac{\tilde{z}_0 - z_0}{z_0}N + o(1)\Big)\frac{\mathrm{Det}[1 - z_0 Q_0 G_L Q_0]}{\mathrm{Det}[1 - \tilde{z}_0 Q_0 G_L Q_0]} \qquad (3.22)$$
$$\times\frac{\mathrm{Det}[1 - \tilde{z}_0 Q_0 G_L Q_0]}{\mathrm{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]}.$$

Lemma 3.9 *Under the thermodynamic limit, it holds that*

$$\text{(i)} \qquad \frac{\mathrm{Det}[1 - z_0 Q_0 G_L Q_0]}{\mathrm{Det}[1 - \tilde{z}_0 Q_0 G_L Q_0]} = \exp\Big(\frac{\tilde{z}_0 - z_0}{z_0}(N - p_0^{(N)}) + o(1)\Big),$$

$$\text{(ii)} \qquad \frac{\mathrm{Det}[1 - \tilde{z}_0 Q_0 \tilde{G}_L Q_0]}{\mathrm{Det}[1 - \tilde{z}_0 Q_0 G_L Q_0]} = \mathrm{Det}[1 + K_f](1+o(1)).$$

Proof : Put $h(z) = -\log\mathrm{Det}(1 - zQ_0G_LQ_0) = -\sum_{j=1}^{\infty}\log(1 - zg_j)$, and we have

$$\log\left(\frac{\mathrm{Det}[1 - z_0 Q_0 G_L Q_0]}{\mathrm{Det}[1 - \tilde{z}_0 Q_0 G_L Q_0]}\right) = h(\tilde{z}_0) - h(z_0) = h'(z_0)(\tilde{z}_0 - z_0) + \frac{1}{2}h''(\bar{z}_0)(\tilde{z}_0 - z_0)^2,$$

where $\bar{z}_0 \in (z_0, \tilde{z}_0)$. Hence we get (i) by

$$h'(z_0) = \sum_{j=1}^{\infty}\frac{g_j}{1 - z_0 g_j} = \frac{N - p_0}{z_0}$$

and

$$h''(\bar{z}_0)(\tilde{z}_0 - z_0)^2 = \sum_{j=1}^{\infty}\frac{g_j^2(\tilde{z}_0 - z_0)^2}{(1 - \bar{z}_0 g_j)^2} \leqslant \sum_{j=1}^{\infty}\frac{g_j(\tilde{z}_0 - z_0)^2}{(1 - g_j)^2} = O(L^{-2d}\ell(L)) = o(1),$$

where Lemma 3.3(i) has been used.

(ii) Thanks to the product and cyclic properties of the Fredholm determinant, we have

$$\frac{\mathrm{Det}[1-\tilde{z}_0 Q_0 \tilde{G}_L Q_0]}{\mathrm{Det}[1-\tilde{z}_0 Q_0 G_L Q_0]} = \mathrm{Det}[1+\tilde{z}_0 Q_0 (G_L - \tilde{G}_L) Q_0 (1-\tilde{z}_0 Q_0 G_L Q_0)^{-1}]$$

$$= \mathrm{Det}[1+\tilde{z}_0 \sqrt{1-e^{-f}} Q_0 G_L Q_0 (1-\tilde{z}_0 Q_0 G_L Q_0)^{-1} \sqrt{1-e^{-f}}].$$

Note that $L^2(\Lambda_L)$ can be identified with an closed subspace of $L^2(\mathbb{R}^d)$ naturally. By this identification, we regard G_L and $\sqrt{1-e^{-f}}$ as operators on $L^2(\mathbb{R}^d)$. It is enough to prove

$$A_L = \tilde{z}_0 \sqrt{1-e^{-f}} Q_0 G_L Q_0 (1-\tilde{z}_0 Q_0 G_L Q_0)^{-1} \sqrt{1-e^{-f}} \longrightarrow K_f$$

in the trace norm. In the following, we show $A_L \to K_f$ strongly and $||A_L||_T \to ||K_f||_T$. Then the Grüm's convergence theorem [5] yields the above.

For $\psi, \phi \in L^2(\mathbb{R}^d)$, we have

$$|(\psi, (A_L - K_f)\phi)| = \Big| \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \overline{\psi(x)} \sqrt{1-e^{-f(x)}} \phi(y) \sqrt{1-e^{-f(y)}}$$

$$\times \Big(\tilde{z}_0 [Q_0 G_L Q_0 (1-\tilde{z}_0 Q_0 G_L Q_0)^{-1}](x,y) - K(x,y) \Big) \Big| \tag{3.23}$$

$$\leqslant ||\psi||_2 ||\phi||_2 ||\sqrt{1-e^{-f}}||_2^2 \sup_{x,y \in \mathrm{supp}\, f} |\tilde{z}_0 [Q_0 G_L Q_0 (1-\tilde{z}_0 Q_0 G_L Q_0)^{-1}](x,y) - K(x,y)|,$$

which tends to 0, by Lemma 3.3(iv). Note that it might be possible that $\tilde{z}_0 = \tilde{z}_0(L,N) > 1$ holds in the course of the thermodynamic limit. However, $\tilde{z}_0^{-1}$ is well separated from $\mathrm{Spec}\, Q_0 G_L Q_0$ since $|1-\tilde{z}_0| = O(L^{-d})$. Hence, the proof of Lemma 3.3(iv) is still valid in this case. Thus the strong (in fact the norm) convergence has been proved. For the convergence of the trace norm, we use Lemma 3.3(iv) again and positive self-adjointness of operators A_L and K_f to get

$$||A_L||_T - ||K_f||_T = \mathrm{Tr}[A_L - K_f]$$

$$= \int_{\mathbb{R}^d} dx\, (1-e^{-f(x)}) (\tilde{z}_0 [Q_0 G_L Q_0 (1-\tilde{z}_0 Q_0 G_L Q_0)^{-1}](x,x) - K(x,x)) \to 0. \quad \square$$

Together with Lemma 3.4(i) and Lemma 3.7(i,ii), we get the formula

$$\mathrm{E}^B_{L,N}\Big[e^{-<f,\xi>}\Big] = (1+o(1)) \times \tag{3.24}$$

$$\frac{\exp\Big(-(\rho-\rho_c)(\sqrt{1-e^{-f}}, [1+W_L^* Q_0 (1-Q_0 G_L Q_0)^{-1} Q_0 W_L]^{-1} \sqrt{1-e^{-f}})\Big)}{\mathrm{Det}[1+K_f]}.$$

From the convergence $W_L^* Q_0 (1-Q_0 G_L Q_0)^{-1} Q_0 W_L = A_L \to K_f$ in the thermodynamic limit, we have proved the theorem.

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A Complex integrals

Lemma A.1 *For $0 \leqslant x \leqslant 1$ and $p \geqslant 0$ satisfying $0 \leqslant px < 1$, it holds that*

$$1 \geqslant (1+x)^p(1-px) \geqslant \exp\Big(-\frac{p(1+p)(1+px^2)}{2(1-px)^2}x^2\Big).$$

Proof: Put $f(x) = \log[(1+x)^p(1-px)]$, then

$$f'(x) = \frac{p}{1+x} - \frac{p}{1-px}, \qquad f''(x) = -\frac{p(1+p)(1+px^2)}{(1+x)^2(1-px)^2}$$

hold. So we have $f(0) = 0,\ f'(0) = 0$ and $0 \geqslant f''(\theta x) \geqslant -p(1+p)$ $(1+px^2)/(1-px)^2$ for $\theta \in (0,1)$, which imply the result. $\square$

Lemma A.2 *Let the collection of numbers $\{\, p_j^{(N)}\,\}_{j,N}$ satisfy*

$$p_0^{(N)} > p_1^{(N)} \geqslant p_2^{(N)} \geqslant \cdots \geqslant p_j^{(N)} \geqslant \cdots \geqslant 0, \quad \sum_{j=0}^{\infty} p_j^{(N)} = N.$$

Suppose that there exist a sequence $\{\, R^{(N)}\}_{N\in\mathbb{N}}$ and $c \in (0,1)$ such that

$$1 < R^{(N)} < cp_0^{(N)}(1 \wedge p_1^{(N)-1}), \quad \lim_{N\to\infty} p_0^{(N)}/R^{(N)}e^{c'R^{(N)}} = 0$$

and

$$\lim_{N\to\infty} R^{(N)2}\sum_{j=1}^{\infty} p_j^{(N)}(1+p_j^{(N)})p_0^{(N)-2} = 0,$$

where $c' = c^{-1}\log(1+c)$. Then

$$\lim_{N\to\infty} p_0^{(N)} \oint_{S_1(0)} \frac{d\eta}{2\pi i}\frac{1}{\eta^{N+1}\prod_{j=0}^{\infty}(1-p_j^{(N)}(\eta-1))} = \frac{1}{e}$$

holds.

Proof: We omit the superscript (N) here. Note that $p_0 \to \infty$ and $R \to \infty$ as $N \to \infty$. By the preceding lemma,

$$1 \geqslant \prod_{j=1}^{\infty} \left[\left(1 + \frac{R}{p_0}\right)^{p_j} \left(1 - \frac{Rp_j}{p_0}\right) \right] \geqslant \exp\left(- \sum_{j=1}^{\infty} \frac{p_j(1+p_j)}{2} \frac{(1 + R^2 p_j/p_0^2)R^2}{(1 - Rp_j/p_0)^2 p_0^2} \right).$$

So the assumption on R implies

$$\prod_{j=1}^{\infty} \left[\left(1 + \frac{R}{p_0}\right)^{p_j} \left(1 - \frac{Rp_j}{p_0}\right) \right] \underset{N\to\infty}{\longrightarrow} 1.$$

Similarly, we have

$$\prod_{j=1}^{\infty} \left[\left(1 + \frac{1}{p_0}\right)^{p_j} \left(1 - \frac{p_j}{p_0}\right) \right] \underset{N\to\infty}{\longrightarrow} 1.$$

Now let us deform the integration contour of η to two parts

$$\oint_{S_1(0)} = - \oint_{S_{(R-1)/p_0}(1+1/p_0)} + \oint_{S_{1+R/p_0}(0)} = I_1 + I_2.$$

I_1 is obtained by the residue at $\eta = 1 + 1/p_0$:

$$I_1 = -p_0 \left[\left(1 + \frac{1}{p_0}\right)^{N+1} (-p_0) \prod_{j=1}^{\infty} \left(1 - \frac{p_j}{p_0}\right) \right]^{-1}$$

$$= \left(1 + \frac{1}{p_0}\right)^{-p_0 - 1} \prod_{j=1}^{\infty} \left[\left(1 + \frac{1}{p_0}\right)^{p_j} \left(1 - \frac{p_j}{p_0}\right) \right]^{-1} \underset{N\to\infty}{\longrightarrow} e^{-1}.$$

I_2 can be estimated as

$$|I_2| \quad \leqslant \quad p_0 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \prod_{j=0}^{\infty} \left[\left(1 + \frac{R}{p_0}\right)^{p_j} \left| 1 - p_j \left(\left(1 + \frac{R}{p_0}\right) e^{i\theta} - 1 \right) \right| \right]^{-1}$$

$$\leqslant p_0 \left(1 + \frac{R}{p_0}\right)^{-p_0} |\, 1 - R|^{-1} \left[\prod_{j=1}^{\infty} \left(1 + \frac{R}{p_0}\right)^{p_j} \left(1 - \frac{Rp_j}{p_0}\right) \right]^{-1} \underset{N\to\infty}{\longrightarrow} 0,$$

since $(1 + R/p_0)^{p_0} \geqslant (1 + c)^{R/c} = e^{c'R}$ and the assumption. □

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