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Path Coloring on Binary Caterpillars*

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SUMMARY The path coloring problem is to assign the minimum number of colors to a given set $\mathcal{P}$ of directed paths on a given symmetric digraph D so that no two paths sharing an arc have the same color. The problem has applications to efficient assignment of wavelengths to communications on WDM optical networks. In this paper, we show that the path coloring problem is NP-hard even if the underlying graph of D is restricted to a binary caterpillar. Moreover, we give a polynomial time algorithm which constructs, given a binary caterpillar G and a set $\mathcal{P}$ of directed paths on the symmetric digraph associated with G, a path coloring of $\mathcal{P}$ with at most $\lceil \frac{8}{5}L \rceil$ colors, where L is the maximum number of paths sharing an edge. Furthermore, we show that no local greedy path coloring algorithm on caterpillars in general uses less than $\lfloor \frac{8}{5}L \rfloor$ colors.
key words: *path coloring, wavelength routing, caterpillar*

1. Introduction

The *wavelength routing problem* is, given a set of ordered pairs of nodes on a symmetric digraph, to construct directed paths connecting the given pairs of nodes and to assign the minimum number of colors to the constructed paths so that no two paths sharing an arc have the same color. The problem has applications to efficient assignment of wavelengths to communications on WDM (wavelength division multiplexing) optical networks. The problem has a variant called the *path coloring problem*, in which directed paths instead of ordered pairs of nodes are given as the input, i.e., the paths are predefined. The path coloring problem is a natural formulation if the underlying graph of the given graph is a tree because the path connecting a pair of nodes is uniquely determined in a tree. In this paper, we focus on the (directed) path coloring problem. A survey on the wavelength routing problem and the path coloring problem including some variants of these problems, such as undirected versions of the problems, can be found in, e.g., [3]. Every digraph considered here is symmetric, and hence, can be determined uniquely

from its underlying undirected graph. In what follows, for an undirected graph G, the path coloring on the symmetric digraph associated with G is also called the path coloring on G for simplicity.

The path coloring problem has been studied extensively so far. The problem is NP-hard even if the underlying graph G of the given digraph is restricted to a binary tree [12][6], to a tree of depth 2 or 3 [12][6], or to a ring [9]. Moreover, there exists no polynomial-time algorithm with approximation ratio less than $\frac{4}{3}$ even if G is restricted to a tree, unless P=NP [6]. However, it is known that if G is restricted to a path or to a star, which is defined as a tree with a unique vertex of degree more than 1, then there exists a path coloring algorithm which uses L colors [2][10], where L is the *(edge) congestion* of the given set $\mathcal{P}$ of directed paths. It should be noted that L is a lower bound of the number of colors to be assigned to $\mathcal{P}$. Therefore, the above algorithms for paths and stars are optimal. For general trees, Kaklamanis, Persiano, Erlebach, and Jansen [7] gave a *local greedy* path coloring algorithm on trees with at most $\lceil \frac{5}{3}L \rceil$ colors. A local greedy path coloring algorithm on a tree is an algorithm which colors the paths in the order of the depth-first search on the tree without information on unsearched arcs of the paths, and never recolors the paths already colored. It is shown in [11][7] that no local greedy path coloring algorithm on trees in general uses less than $\lfloor \frac{5}{3}L \rfloor$ colors. As for the general lower bound for trees, it is known that there exist a binary tree with exactly two vertices of degree 3 and a set of directed paths on the tree such that at least $\frac{5}{4}L$ colors is necessary to color the paths [14]. A probabilistic path coloring on binary trees was shown in [1]. Other networks than trees, such as rings and trees of rings, were investigated in, e.g., [16][15][13][4][5].

In this paper, we consider the path coloring problem on caterpillars, which are defined as trees whose vertices of degree more than 1 are contained in a single path. It is interesting to study the path coloring problem on caterpillars because caterpillars are the simplest trees in the sense that they are close to, but different from paths. Besides, the class of caterpillars contains important classes of trees such as paths and stars, for which optimal algorithms are known, and the tree which gives the lower bound of $\frac{5}{4}L$. Therefore, considering the path coloring problem on caterpillars can

Fig. 1 An example of a caterpillar.

Fig. 2 An example of R and $\mathcal{Q}$ for an instance of CIRCULAR ARC COLORING.
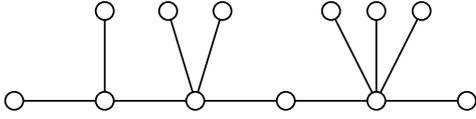
be a fundamental step to close the gap between upper and lower bounds in trees. Caterpillars are investigated also in [8] for the minimum collisions path multicoloring problem, which is another variant of the path coloring problem.

In this paper, we show that the path coloring problem is NP-hard even if the underlying graph of the given digraph is restricted to a binary caterpillar. Moreover, we give an (essentially) local greedy polynomial time path coloring algorithm on binary caterpillars with at most $\lceil \frac{8}{5}L \rceil$ colors. Furthermore, we show that no local greedy path coloring algorithm on binary caterpillars in general uses less than $\lfloor \frac{8}{5}L \rfloor$ colors.

The paper is organized as follows: Some definitions are given in Sect. 2. In Sect. 3, we show the NP-hardness of the path coloring problem on binary caterpillars. We give our path coloring algorithm for binary caterpillars in Sect. 4. We show the lower bound for local greedy path coloring algorithms on caterpillars in Sect. 5. We conclude the paper with some remarks in Sect. 6.

2. Preliminaries

The *symmetric digraph* D associated with an (undirected) graph G is the graph obtained from G by replacing each edge e of G with two oppositely oriented arcs with the same ends as e. G is called the *underlying graph* of D. In what follows, we identify a graph with its associated symmetric digraph for simplicity, i.e., we regard an (undirected) edge (u, v) as a pair of arcs (u, v) and (v, u), and vice versa.

A tree is said to be *binary* if its maximum vertex degree is at most 3. A *caterpillar* is a tree whose vertices of degree more than 1 are contained in a single path (Fig. 1). A maximal path containing all the vertices of degree more than 1 is called a *spine*.

Two or more directed paths are said to *intersect* if all the paths share an arc. The *(edge-)congestion* of a set $\mathcal{P}$ of directed paths is the maximum number of intersecting paths of $\mathcal{P}$. A *path coloring*, or simply, a *coloring* of $\mathcal{P}$ is an assignment of colors to the paths of $\mathcal{P}$ such that no two intersecting paths of $\mathcal{P}$ have the same color.

3. NP-Hardness

In this section, we show the NP-hardness of the path coloring problem on binary caterpillars by proving the following theorem:
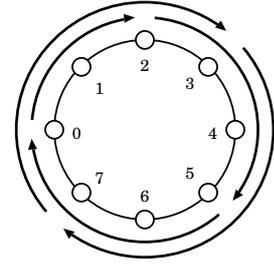
Theorem 1: The problem of determining, given a binary caterpillar G, a set $\mathcal{P}$ of directed paths on G, and an integer k, whether $\mathcal{P}$ can be colored with at most k colors is NP-complete.

The main idea of our proof is essentially same as the proof of [6] for NP-hardness on binary trees. We prove Theorem 1 by constructing a polynomial reduction from CIRCULAR ARC COLORING, which is known to be NP-complete [9] and can be defined in the context of the path coloring problem as follows:

CIRCULAR ARC COLORING

Instance A ring R with the vertex set $\{0, \ldots, n-1\}$ ($n \geq 3$) and edge set $\{(i, (i+1) \bmod n) \mid 0 \leq i < n\}$, a set $\mathcal{Q}$ of directed paths consisting of arcs $(i, (i+1) \bmod n)$ ($0 \leq i < n$) on R, and an integer $k \leq |Q|$.
Question Does there exist a coloring of $\mathcal{Q}$ with at most k colors?

Figure 2 shows an example of R and $\mathcal{Q}$.

3.1 Translation of Instance

For a ring R, a set $\mathcal{Q}$ of directed paths on R, and an integer $k \leq |\mathcal{Q}|$ given as an instance of CIRCULAR ARC COLORING, we construct a binary caterpillar G and a set $\mathcal{P}$ of directed paths on G as follows:

1. Suppose that $\mathcal{Q} = \{Q_0, \ldots, Q_{|\mathcal{Q}|-1}\}$ such that the paths containing the arc $(n-1, 0)$ are Q_0, …, Q_{r-1}.
2. Let G be the caterpillar with the vertex set V and edge set E defined as follows:

$$V = \{u_0, \ldots, u_{n-1}\} \cup \{s_0, \ldots, s_r\} \cup \{s'_0, \ldots, s'_r\}$$
$$\cup \{t_0, \ldots, t_r\} \cup \{t'_0, \ldots, t'_r\},$$
$$E = \{(u_i, u_{i+1}) \mid 0 \leq i < n-1\}$$
$$\cup \{(s_i, s_{i+1}) \mid 0 \leq i < r\} \cup \{(s_i, s'_i) \mid 0 \leq i \leq r\}$$
$$\cup \{(t_i, t_{i+1}) \mid 0 \leq i < r\} \cup \{(t_i, t'_i) \mid 0 \leq i \leq r\}$$
$$\cup \{(u_{n-1}, s_r), (u_0, t_r)\}.$$

3. For $r \leq h < |\mathcal{Q}|$, let P_h be the directed path from u_i to u_j on G, where i and j ($i < j$) are the end-vertices of Q_h.
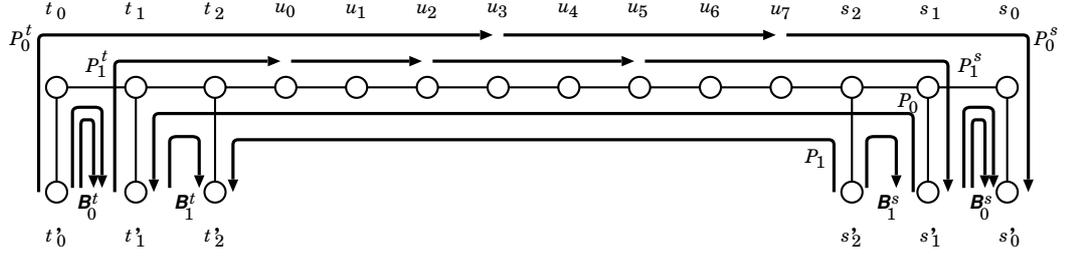
Fig. 3 G and $\mathcal{P}$ translated from the example of R and $\mathcal{Q}$ of Fig. 2 for the case that $k = 3$, and Q_0 and Q_1 are the directed paths on R from 7 to 3 and from 5 to 0, respectively.

4. For $0 \leq h < r$, let P_h^s, P_h, and P_h^t be the directed paths on G from u_j to s'_h, from s'_{h+1} to t'_{h+1}, and from t'_h to u_i, respectively, where i and j $(i < j)$ are the end-vertices of Q_h. Moreover, let $\mathcal{B}_h^s$ and $\mathcal{B}_h^t$ be the sets of $k - h - 1$ directed paths on G from s'_{h+1} to s'_h and from t'_h to t'_{h+1}, respectively.

5. Let $\mathcal{P} = \{P_h \mid 0 \leq h < |\mathcal{Q}|\} \cup \{P_h^s \mid 0 \leq h < r\} \cup \{P_h^t \mid 0 \leq h < r\} \cup \bigcup_{0 \leq h < r}(\mathcal{B}_h^s \cup \mathcal{B}_h^t)$.

Figure 3 shows G and $\mathcal{P}$ translated from the example of R and $\mathcal{Q}$ of Fig. 2.

3.2 Correspondence of Answers

We prove that $\mathcal{Q}$ can be colored with at most k colors if and only if $\mathcal{P}$ can be colored with at most k colors.

The following lemma can be observed by the definitions of $\mathcal{Q}$, G, and $\mathcal{P}$:

Lemma 1: A pair of paths of $\mathcal{P}$ intersect if and only if it is one of the following pairs of paths:

(a) P_h and P_l with $r \leq h < l < |\mathcal{Q}|$ such that Q_h and Q_l intersect.
(b) P_h^s and P_l, and/or P_h^t and P_l with $0 \leq h < r \leq l < |\mathcal{Q}|$ such that Q_h and Q_l intersect.
(c) Two paths of P_0^s, ..., P_{r-1}^s.
(d) Two paths of P_0, ..., P_{r-1}.
(e) Two paths of P_0^t, ..., P_{r-1}^t.
(f) A path of P_0^s, ..., P_h^s and a path of $\mathcal{B}_h^s$ with $0 \leq h < r$.
(g) A path of P_0^t, ..., P_h^t and a path of $\mathcal{B}_h^t$ with $0 \leq h < r$.
(h) P_h and a path of $\mathcal{B}_h^s \cup \mathcal{B}_h^t$ with $0 \leq h < r$.
$\square$

Lemma 2: $\mathcal{P}$ can be colored with at most k colors if $\mathcal{Q}$ can be colored with at most k colors.

Proof Assume that there exists a coloring of $\mathcal{Q}$ with a set C of at most k colors. If $|C| < k$, then we add dummy colors to C so that $|C| = k$. We define an assignment of C to $\mathcal{P}$ as follows:

1. For $r \leq h < |\mathcal{Q}|$, assign the color of Q_h to P_h.
2. For $0 \leq h < r$, assign the color of Q_h to P_h^s, P_h, and P_h^t.

3. For $0 \leq h < r$, assign the distinct colors of C which are not assigned to P_0, ..., P_h to the paths of $\mathcal{B}_h^s$ and to those of $\mathcal{B}_h^t$.

Since $|C| = k$ and $|\mathcal{B}_h^s| = |\mathcal{B}_h^t| = k - h - 1$, it should be noted that the assignment of 3 is well-defined. Thus, it suffices to show that no two intersecting paths of $\mathcal{P}$ have the same color. It follows from Lemma 1 that any pair of intersecting paths is one of the pairs in (a) through (h) of Lemma 1. A pair in (a) or (b) has distinct colors by the assignments of 1 and 2. A pair in (c), (d), or (e) has distinct colors by the assignment of 2 and by the fact that Q_0, ..., Q_{r-1} intersect. A pair in (f), (g), or (h) has distinct colors by the assignments of 2 and 3. Thus, we have the lemma. $\square$

Lemma 3: If there exists a coloring of $\mathcal{P}$ with at most k colors, then P_h^s, P_h, and P_h^t have the same color for $0 \leq h < r$.

Proof Assume that there exists a coloring of $\mathcal{P}$ with a set C of at most k colors. We prove by induction on h that P_h^s and P_h have the same color.

It follows from the definitions of G and $\mathcal{P}$ that P_0^s and the paths of $\mathcal{B}_0^s$ intersect at (s_1, s_0) and that P_0 and the paths of $\mathcal{B}_0^s$ intersect at (s'_1, s_1). Since $|\mathcal{B}_0^s| = k - 1$ and $|C| = k$, exactly one color of C which is not assigned to a path of $\mathcal{B}_0^s$ can be assigned to P_0^s and P_0. Thus, P_0^s and P_0 have the same color.

Assume that $P_{h'}^s$ and $P_{h'}$ have the same color for $h' < h$ $(0 < h < r)$. It follows from the definitions of G and $\mathcal{P}$ that P_0^s, ..., P_h^s and the paths of $\mathcal{B}_h^s$ intersect at (s_{h+1}, s_h) and that P_h and the paths of $\mathcal{B}_h^s$ intersect at (s'_{h+1}, s_{h+1}). Since $|\mathcal{B}_h^s| = k - h - 1$, P_h has one of the colors of P_0^s, ..., P_h^s. However, P_h cannot have the colors of P_0^s, ..., P_{h-1}^s because P_0, ..., P_h intersect by the definitions of G and $\mathcal{P}$, and $P_{h'}^s$ and $P_{h'}$ have the same color for $h' < h$ by induction hypothesis. Thus, P_h^s and P_h have the same color.

We can show by a similar argument that P_h^t and P_h have the same color for $0 \leq h < r$, and hence omit the proof. $\square$

Lemma 4: $\mathcal{Q}$ can be colored with at most k colors if $\mathcal{P}$ can be colored with at most k colors.

Proof Assume that there exists a coloring of $\mathcal{P}$ with a set C of at most k colors. We define the assignment of C to $\mathcal{Q}$ so that Q_h has the color of P_h for $0 \le h < |\mathcal{Q}|$. The assignment is a coloring of $\mathcal{Q}$ because P_h and P_l intersect if and only if Q_h and Q_l intersect for $0 \le h < l < |\mathcal{Q}|$ by Lemmas 1 and 3. □

G has the size $O(n + r) = O(n + |\mathcal{Q}|)$, and $\mathcal{P}$ has the size $O(|\mathcal{Q}| + r + kr) = O(|\mathcal{Q}|^2)$. Thus, we have obtained a desired polynomial reduction by Lemmas 2 and 4. Therefore, the proof of Theorem 1 is completed.

4. Path Coloring Algorithm

In this section, we show the following theorem by constructing a desired algorithm:

Theorem 2: There exists a polynomial time path coloring algorithm on caterpillars with at most $\lceil \frac{8}{5}L \rceil$ colors, where L is the congestion of a given set of directed paths.

4.1 Definition

We describe our algorithm `PC_main`, together with a sub-routine `Assign`.

`Assign($\mathcal{P}'$, $C'$)`

Given a set $\mathcal{P}'$ of directed paths on G and a set C' of colors, the algorithm `Assign` assigns distinct colors of C' to paths of $\mathcal{P}'$ as follows:

- If $|\mathcal{P}'| \le |C'|$, then arbitrarily selected $|\mathcal{P}'|$ colors of C' are assigned to all the paths of $\mathcal{P}'$.
- If $|\mathcal{P}'| > |C'|$, then all the colors of C' are assigned to arbitrarily selected $|C'|$ paths of $\mathcal{P}'$. (Not selected paths of $\mathcal{P}'$ are left uncolored.)

`PC_main($G$, $\mathcal{P}$)`

Given a binary caterpillar G and a set $\mathcal{P}$ of directed paths on G, the algorithm `PC_main` colors $\mathcal{P}$ as follows:

1. Suppose that the spine of G has vertices u_0, ..., u_n and edges (u_0, u_1), ..., (u_{n-1}, u_n).
2. Let C be a set of $\lceil \frac{8}{5}L \rceil$ colors, where L is the congestion of $\mathcal{P}$.
3. Let $\overline{\mathcal{P}}$ be the set of paths of $\mathcal{P}$ with at least two edges.
4. Execute the following steps for $i = 0$ to $n - 1$:
   a. If $i = 0$, then let $\mathcal{P}_3 = \mathcal{P}_4 = \emptyset$. Otherwise, let $\mathcal{P}_3$ and $\mathcal{P}_4$ be the sets of paths passing from u_{i-1} to u_{i+1} and from u_{i+1} to u_{i-1}, respectively.
   b. Let C_3 and C_4 be the sets of colors of $\mathcal{P}_3$ and $\mathcal{P}_4$, respectively, and let $C_5 = C_6 = \emptyset$.
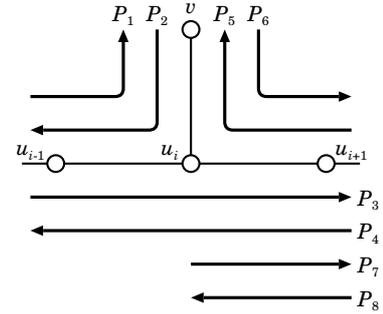   c. If u_i is adjacent to a vertex v which is not con-

Fig. 4 Paths $\mathcal{P}_1$, ..., $\mathcal{P}_8$ around u_i.

tained in the spine, then execute the following steps:

   i. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the sets of paths passing from u_{i-1} to v and from v to u_{i-1}, respectively.
   ii. Let C_1 and C_2 be the sets of colors of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.
   iii. Let $\mathcal{P}_5$ and $\mathcal{P}_6$ be the sets of paths passing from u_{i+1} to v and from v to u_{i+1}, respectively.
   iv. Execute `Assign($\mathcal{P}_5$, $C_3 - C_4$)` and `Assign($\mathcal{P}_6$, $C_4 - C_3$)`.
   v. Let $\mathcal{P}_5'$ and $\mathcal{P}_6'$ be the sets of the uncolored paths of $\mathcal{P}_5$ and $\mathcal{P}_6$, respectively.
   vi. Let C_{56}' be a set of $\min\{\max\{|\mathcal{P}_5'|, |\mathcal{P}_6'|\}, |C - \bigcup_{1 \le j \le 4} C_j|\}$ colors of $C - \bigcup_{1 \le j \le 4} C_j$.
   vii. Execute `Assign($\mathcal{P}_5'$, $C_{56}'$)` and `Assign($\mathcal{P}_6'$, $C_{56}'$)`.
   viii. Let $\mathcal{P}_5''$ and $\mathcal{P}_6''$ be the sets of the uncolored paths of $\mathcal{P}_5'$ and $\mathcal{P}_6'$, and execute `Assign($\mathcal{P}_5''$, $C_2 - \bigcup_{j=1,3,4} C_j$)` and `Assign($\mathcal{P}_6''$, $C_1 - \bigcup_{j=2,3,4} C_j$)`.
   ix. Let C_5 and C_6 be the set of colors of $\mathcal{P}_5$ and $\mathcal{P}_6$, respectively.
   d. Let $C_{78}' \supseteq \bigcup_{3 \le j \le 6} C_j$ be a set of $\max\{L, |\bigcup_{3 \le j \le 6} C_j|\}$ colors of C.
   e. For the set $\mathcal{P}_7$ of paths of $\overline{\mathcal{P}}$ which start at u_i and contain u_{i+1}, execute `Assign($\mathcal{P}_7$, $C_{78}' - (C_3 \cup C_6)$)`. Similarly, for the set $\mathcal{P}_8$ of paths of $\overline{\mathcal{P}}$ which end at u_i and contain u_{i+1}, execute `Assign($\mathcal{P}_8$, $C_{78}' - (C_4 \cup C_5)$)`.

5. For each $P \in \mathcal{P} - \overline{\mathcal{P}}$, assign P a color c of C which has not been assigned to the paths intersecting with P.

Figure 4 shows the paths $\mathcal{P}_1$, ..., $\mathcal{P}_8$ around u_i. Our algorithm can be clearly executed in polynomial time.

4.2 Correctness

To prove Theorem 2, it suffices to show that `PC_main` is

well-defined, i.e., all the paths of $\mathcal{P}$ are assigned colors of C so that no two intersecting paths have the same color. This is because C is defined as a set of $\lceil \frac{8}{5}L \rceil$ colors.

Lemma 5: For $1 \leq i < n$, if $\bigcup_{1 \leq j \leq 4} \mathcal{P}_j$ has been colored with at most $\frac{6}{5}L$ colors so far, then Step 4 assigns all the paths of $\mathcal{P}_5$ and $\mathcal{P}_6$ distinct colors of $C - (C_1 \cup C_4)$ and $C - (C_2 \cup C_3)$, respectively.

Proof We can observe by the definition of Step 4 that $\mathcal{P}_5$ has colors of $C - (C_1 \cup C_4)$ and that all the colors of $C - (C_1 \cup C_4)$ are assigned to $\mathcal{P}_5$ while the uncolored paths of $\mathcal{P}_5$ remain in Step 4(c)viii. Similarly $\mathcal{P}_6$ has colors of $C - (C_2 \cup C_3)$ and that all the colors of $C - (C_2 \cup C_3)$ are assigned to $\mathcal{P}_6$ while the uncolored paths of $\mathcal{P}_6$ remain in Step 4(c)viii. Thus, it suffices to show that $|\mathcal{P}_5| \leq |C - (C_1 \cup C_4)|$ and $|\mathcal{P}_6| \leq |C - (C_2 \cup C_3)|$.

Since the paths of $\mathcal{P}_1 \cup \mathcal{P}_5$ intersect and so do the paths of $\mathcal{P}_4 \cup \mathcal{P}_5$, we have that $|\mathcal{P}_5| \leq L - \max\{|\mathcal{P}_1|, |\mathcal{P}_4|\} = L - \max\{|C_1|, |C_4|\} \leq L - \frac{|C_1| + |C_4|}{2} \leq L - \frac{|C_1 \cup C_4|}{2}$. Moreover, $|C_1 \cup C_4| \leq |\bigcup_{1 \leq j \leq 4} C_j| \leq \frac{6}{5}L$ by the assumption of the lemma. Thus, it follows that $|C - (C_1 \cup C_4)| - |\mathcal{P}_5| \geq |C| - |C_1 \cup C_4| - (L - \frac{|C_1 \cup C_4|}{2}) \geq \lceil \frac{8}{5}L \rceil - L - \frac{3}{5}L \geq 0$. The proof of $|C - (C_2 \cup C_3)| - |\mathcal{P}_6| \geq 0$ can be accomplished by a similar argument, and is omitted. □

Lemma 6: For $1 \leq i < n$, if $\bigcup_{1 \leq j \leq 4} \mathcal{P}_j$ has been colored with at most $\frac{6}{5}L$ colors so far, then Step 4 assigns $\mathcal{P}_5$ and $\mathcal{P}_6$ colors so that $|\bigcup_{3 \leq j \leq 6} C_j| \leq \frac{6}{5}L$.

Proof Let $C_{45} = C_4 \cup C_5$ and $C_{36} = C_3 \cup C_6$. Since the paths of $\mathcal{P}_4 \cup \mathcal{P}_5$ intersect and so do the paths of $\mathcal{P}_3 \cup \mathcal{P}_6$, it follows that $|C_{45}| \leq L$ and $|C_{36}| \leq L$.

We first consider the case that $\mathcal{P}_5' = \mathcal{P}_6' = \emptyset$. Then all the paths of $\mathcal{P}_5 \cup \mathcal{P}_6$ obtain colors in Step 4(c)iv. Thus, we have by the definition of Step 4(c)iv that $C_{45} \subseteq (C_3 - C_4) \cup C_4 = C_3 \cup C_4$ and $C_{36} \subseteq (C_4 - C_3) \cup C_3 = C_3 \cup C_4$. Moreover, we have by the assumption of the lemma that $|\bigcup_{1 \leq j \leq 4} C_j| \leq \frac{6}{5}L$. Therefore, it follows that $|\bigcup_{3 \leq j \leq 6} C_j| = |C_{45} \cup C_{36}| \leq |C_3 \cup C_4| \leq |\bigcup_{1 \leq j \leq 4} C_j| \leq \frac{6}{5}L$.

We next consider the case that $\mathcal{P}_5' \neq \emptyset$, $|\mathcal{P}_5'| \geq |\mathcal{P}_6'|$, and $|\mathcal{P}_6''| = \emptyset$. Then there exists a path of $\mathcal{P}_5$ obtaining a color in Step 4(c)vii, and no path of $\mathcal{P}_6$ obtains a color in Step 4(c)viii. Since $|\mathcal{P}_5'| \geq |\mathcal{P}_6'|$, we have that $C_{45} \supseteq (C_3 - C_4) \cup C_4 \cup C_{56}' = C_3 \cup C_4 \cup C_{56}'$ and that $C_6 \subseteq (C_4 - C_3) \cup C_{56}' \subseteq C_{45}$. Thus, it follows that $|\bigcup_{3 \leq j \leq 6} C_j| \leq |C_{45}| \leq L \leq \frac{6}{5}L$.

The proof for the case that $\mathcal{P}_6' \neq \emptyset$, $|\mathcal{P}_6'| \geq |\mathcal{P}_5'|$, and $|\mathcal{P}_5''| = \emptyset$ can be accomplished by a similar argument as that for the previous case, and is omitted. Thus, it remains to prove the lemma for the case that $\mathcal{P}_5'' \neq \emptyset$ and $\mathcal{P}_6'' \neq \emptyset$. Then there exist a path of $\mathcal{P}_5$ and a path of $\mathcal{P}_6$ which obtain colors in

Step 4(c)viii. Step 4 assigns colors to all the paths of $\mathcal{P}_5 \cup \mathcal{P}_6$ by Lemma 5. Thus, it follows from the definitions of Steps 4(c)iv, 4(c)vii, and 4(c)viii that $|\bigcup_{3 \leq j \leq 6} C_j| = |C_3 \cup C_4| + |C_{56}'| + |\mathcal{P}_5''| + |\mathcal{P}_6''| = |C_3 \cup C_4| + |\mathcal{P}_5| + |\mathcal{P}_6| - |C_3 - C_4| - |C_4 - C_3| - |C_{56}'| \leq |C_3 \cap C_4| + 2L - \max\{|\mathcal{P}_1|, |\mathcal{P}_4|\} - \max\{|\mathcal{P}_2|, |\mathcal{P}_3|\} - |C| + |\bigcup_{1 \leq j \leq 4} C_j| \leq |C_3 \cap C_4| + \lfloor \frac{2}{5}L \rfloor - \max\{|C_1| + |C_2|, |C_3| + |C_4|\} + |\bigcup_{1 \leq j \leq 4} C_j|$. Thus, if $|C_3 \cup C_4| > \frac{2}{5}L$, then it follows that $|\bigcup_{3 \leq j \leq 6} C_j| \leq |C_3 \cap C_4| + \lfloor \frac{2}{5}L \rfloor - (|C_3| + |C_4|) + |\bigcup_{1 \leq j \leq 4} C_j| = \lfloor \frac{2}{5}L \rfloor - |C_3 \cup C_4| + |\bigcup_{1 \leq j \leq 4} C_j| < \frac{6}{5}L$. On the other hand, if $|C_3 \cup C_4| \leq \frac{2}{5}L$, then it follows that $|\bigcup_{3 \leq j \leq 6} C_j| \leq |C_3 \cap C_4| + \lfloor \frac{2}{5}L \rfloor - (|C_1| + |C_2|) + |\bigcup_{1 \leq j \leq 4} C_j| \leq |C_3 \cap C_4| + \lfloor \frac{2}{5}L \rfloor + |C_3 \cup C_4| \leq \frac{6}{5}L$. □

Lemma 7: For $0 \leq i < n$, if $\bigcup_{3 \leq j \leq 6} \mathcal{P}_j$ has been colored with at most $\frac{6}{5}L$ colors so far, then Step 4e assigns all the paths of $\mathcal{P}_7$ and $\mathcal{P}_8$ distinct colors of $C - (C_3 \cup C_6)$ and $C - (C_4 \cup C_5)$, respectively, so that $|\bigcup_{3 \leq j \leq 8} C_j| \leq \frac{6}{5}L$.

Proof We can observe by the definitions of C_{78}' and Step 4e that $\mathcal{P}_7$ has colors of $C_{78}' - (C_3 \cup C_6) \subseteq C - (C_3 \cup C_6)$. Since the paths of $\mathcal{P}_3 \cup \mathcal{P}_6 \cup \mathcal{P}_7$ intersect, $|\mathcal{P}_3 \cup \mathcal{P}_6 \cup \mathcal{P}_7| \leq L$. Moreover, $|C_{78}'| \geq L$ by definition. Thus, Step 4e assigns all the paths of $\mathcal{P}_7$ distinct colors of $C_{78}' - (C_3 \cup C_6)$. Since $C_{78}' \supseteq \bigcup_{3 \leq j \leq 6} C_j$ by definition, we have that $\bigcup_{3 \leq j \leq 7} C_j \subseteq C_{78}'$. We can also show by a similar argument that Step 4e assigns all the paths distinct colors of $C_{78}' - (C_4 \cup C_5) \subseteq C - (C_4 \cup C_5)$ and that $C_8 \cup \bigcup_{3 \leq j \leq 6} C_j \subseteq C_{78}'$. Since $|C_{78}'| \leq \max\{L, |\bigcup_{3 \leq j \leq 6} C_j|\} \leq \frac{6}{5}L$ by definition and by the assumption of the lemma, we have that $|\bigcup_{3 \leq j \leq 8} C_j| \leq |C_{78}'| \leq \frac{6}{5}L$. □

Lemma 8: Step 4 colors $\overline{\mathcal{P}}$.

Proof For every path $P \in \overline{\mathcal{P}}$, there exists a unique integer i with $0 \leq i < n$ such that $P \in \bigcup_{5 \leq j \leq 8} \mathcal{P}_j$. Let i_P denote such i for P. To prove the lemma, we show by induction on i that for $0 \leq i < n$, Step 4 assigns every path $P \in \overline{\mathcal{P}}$ with $i_P \leq i$ a color which is not assigned to a path $P' \in \overline{\mathcal{P}}$ with $i_{P'} \leq i$ intersecting with P, so that $|\bigcup_{3 \leq j \leq 8} C_j| \leq \frac{6}{5}L$ for i.

We first assume that $i = 0$. Since the spine is a maximal path, every path $P \in \overline{\mathcal{P}}$ with $i_P = 0$ is contained in $\mathcal{P}_7 \cup \mathcal{P}_8$. Moreover, no pair of a path of $\mathcal{P}_7$ and a path of $\mathcal{P}_8$ intersect since G has no cycle. Thus, by Lemma 7, every path $P \in \overline{\mathcal{P}}$ with $i_P = 0$ obtains a color which is not assigned to a path $P' \in \overline{\mathcal{P}}$ with $i_{P'} = 0$ intersecting with P, so that $|\bigcup_{3 \leq j \leq 8} C_j| \leq \frac{6}{5}L$.

We next assume that $i > 0$ and that every path $P \in \overline{\mathcal{P}}$ with $i_P < i$ has obtained a color which is not assigned to a path $P' \in \overline{\mathcal{P}}$ with $i_{P'} < i$ intersecting with P, so that $|\bigcup_{3 \leq j \leq 8} C_j| \leq \frac{6}{5}L$ for $i - 1$, which means $|\bigcup_{1 \leq j \leq 4} C_j| \leq \frac{6}{5}L$ for i. By induction hypothesis and by Lemmas 5, 6, and 7, every

path $P \in \overline{\mathcal{P}}$ with $i_P = i$ obtains a color so that $|\bigcup_{3 \leq j \leq 8} C_j| \leq \frac{6}{5}L$ for i. Let $P, P' \in \overline{\mathcal{P}}$ be intersecting paths with $i_{P'} \leq i_P \leq i$. It should be noted that $P \in \bigcup_{5 \leq j \leq 8} \mathcal{P}_j$ and $P' \in \bigcup_{1 \leq j \leq 8} \mathcal{P}_j$. If $P \in \mathcal{P}_5 \cup \mathcal{P}_6$ and $P' \notin \mathcal{P}_7 \cup \mathcal{P}_8$ then P and P' have distinct colors by induction hypothesis and by Lemma 5. If P and/or P' is contained in $\mathcal{P}_7 \cup \mathcal{P}_8$, then P and P' have distinct colors by induction hypothesis and by Lemma 7.

Therefore, we have the lemma. $\square$

Lemma 9: Step 5 colors $\mathcal{P} - \overline{\mathcal{P}}$ so that any $P \in \mathcal{P} - \overline{\mathcal{P}}$ has a color assigned to no paths intersecting with P.

Proof Since any path $P \in \mathcal{P} - \overline{\mathcal{P}}$ consists of a single edge, there exist at most $L - 1$ paths intersecting with P. Thus, there exists a color in C which is not assigned to a path intersecting with P, and hence Step 5 colors $\mathcal{P} - \overline{\mathcal{P}}$ as desired. $\square$

By Lemmas 8 and 9, our algorithm PC_main colors $\mathcal{P}$ with at most $\lceil \frac{8}{5}L \rceil$ colors. Therefore, the proof of Theorem 2 is completed.

5. Lower Bound

A local greedy path coloring algorithm on a tree is an algorithm which colors the paths in the order of the depth-first search on the tree without information on unsearched arcs of the paths, and never recolors the paths already colored. Thus, our path coloring algorithm given in Sect. 4 is local greedy if all the given paths have length at least 2. However, the paths of length 1 are not essential because, after all the paths of length at least 2 were colored with a color set C by any algorithm, the paths of length 1 can be colored by a straightforward algorithm with $\max\{L, |C|\}$ colors in total, where L is the congestion of the given paths. In fact our lower bound given below still holds even if the given paths are restricted to have length at least 2.

In this section, we show that no local greedy path coloring algorithm on binary caterpillars in general uses less than $\lfloor \frac{8}{5}L \rfloor$ colors. We show this by proving the following theorem:

Theorem 3: For any local greedy path coloring algorithm on caterpillars, there exist a binary caterpillar G and a set $\mathcal{P}$ of directed paths on G such that the algorithm requires at least $\lfloor \frac{8}{5}L \rfloor$ colors to color $\mathcal{P}$, where L is the congestion of $\mathcal{P}$.

We prove the theorem by contradiction, i.e., we assume that there exists a local greedy path coloring algorithm A on caterpillars with at most $\lfloor \frac{8}{5}L \rfloor - 1$ colors for a given set of directed paths with congestion L on the caterpillars, and construct a caterpillar G and a set $\mathcal{P}$ of directed paths with congestion L on G such that A cannot color $\mathcal{P}$ with less than $\lfloor \frac{8}{5}L \rfloor$ colors.

Since A is local greedy and G is a caterpillar, at whichever vertex A begins the depth-first search on G,

the search and hence the path coloring proceeds along the spine of G. Thus, we may assume without loss of generality that the path coloring of A begins at the end-vertex of the spine. Moreover, since A assigns colors to paths without information on unsearched arcs of the paths, we can construct $\mathcal{P}$ just like an online game between an online algorithm and an adversary, i.e., we can determine the structure of the paths passing through a vertex on the spine after A assigned colors to the newly found paths at the previous vertex on the spine.

Lemma 10: Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be two colored sets of L directed paths passing arcs (u, v) and (v, u), respectively. If the total number of colors of paths of $\mathcal{P}_A \cup \mathcal{P}_B$ is βL ($1 \leq \beta \leq 2$), then there exist two sets $\mathcal{P}'_A \subseteq \mathcal{P}_A$ and $\mathcal{P}'_B \subseteq \mathcal{P}_B$ of $\lfloor \frac{\beta L}{2} \rfloor$ paths such that the paths of $\mathcal{P}'_A \cup \mathcal{P}'_B$ have distinct colors.

Proof Let C_A and C_B be the sets of colors of $\mathcal{P}_A$ and $\mathcal{P}_B$, respectively. Since $|\mathcal{P}_A| = |\mathcal{P}_B| = L$, it follows that $|C_A \cap C_B| = |C_A| + |C_B| - |C_A \cup C_B| = (2 - \beta)L$. Thus, there exist two disjoint sets C'_A and C'_B of $L - \lceil \frac{\beta L}{2} \rceil$ colors of $C_A \cap C_B$. We define $\mathcal{P}'_A$ and $\mathcal{P}'_B$ as follows:

- $\mathcal{P}'_A$ is the set of paths of $\mathcal{P}_A$ with colors of $(C_A - C_B) \cup C'_A$.
- $\mathcal{P}'_B$ is the set of paths of $\mathcal{P}_B$ with colors of $(C_B - C_A) \cup C'_B$.

We can observe by definition that the colors of the paths $\mathcal{P}'_A \cup \mathcal{P}'_B$ are distinct. Moreover, we have by definition that $|\mathcal{P}'_A| = |(C_A - C_B) \cup C'_A| = |C_A| - |C_A \cap C_B| + |C'_A| = L - (2 - \beta)L + L - \lceil \frac{\beta L}{2} \rceil = \lfloor \frac{\beta L}{2} \rfloor$ and that $|\mathcal{P}'_B| = |(C_B - C_A) \cup C'_B| = |C_B| - |C_A \cap C_B| + |C'_B| = L - (2 - \beta)L + L - \lceil \frac{\beta L}{2} \rceil = \lfloor \frac{\beta L}{2} \rfloor$ $\square$

Lemma 11: Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be two colored sets of L directed paths passing arcs (u, v) and (v, u), respectively. If the total number of colors of paths of $\mathcal{P}_A \cup \mathcal{P}_B$ is βL ($1 \leq \beta < \frac{3}{2}$), then there exist four disjoint sets $\mathcal{P}_1 \subseteq \mathcal{P}_A$, $\mathcal{P}_2 \subseteq \mathcal{P}_B$, $\mathcal{P}_3 \subseteq \mathcal{P}_A$, and $\mathcal{P}_4 \subseteq \mathcal{P}_B$ of $\lfloor \frac{\beta L}{3} \rfloor$ paths such that the paths of $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ have distinct colors and that $\mathcal{P}_3$ and $\mathcal{P}_4$ have the same color set.

Proof Let C_A and C_B be the sets of colors of $\mathcal{P}_A$ and $\mathcal{P}_B$, respectively. As shown in the proof of Lemma 10, it follows that $|C_A \cap C_B| = (2 - \beta)L$. Thus, there exist two disjoint sets C'_A and C'_B of $L - \lceil \frac{2\beta L}{3} \rceil$ colors of $C_A \cap C_B$. We define $\mathcal{P}_1, \ldots, \mathcal{P}_4$ as follows:

- $\mathcal{P}_1$ is the set of paths of $\mathcal{P}_A$ with colors of $(C_A - C_B) \cup C'_A$.
- $\mathcal{P}_2$ is the set of paths of $\mathcal{P}_B$ with colors of $(C_B - C_A) \cup C'_B$.
- $\mathcal{P}_3$ and $\mathcal{P}_4$ are the sets of $\lfloor \frac{\beta L}{3} \rfloor$ paths of $\mathcal{P}_A$ and $\mathcal{P}_B$, respectively, with the same colors of $(C_A \cap C_B) - (C'_A \cup C'_B)$.

We can observe by definition that $\mathcal{P}_1$, ..., $\mathcal{P}_4$ are disjoint sets with $\mathcal{P}_1 \cup \mathcal{P}_3 \subseteq \mathcal{P}_A$ and $\mathcal{P}_2 \cup \mathcal{P}_4 \subseteq \mathcal{P}_B$; that the paths of $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ have distinct colors; and that $\mathcal{P}_3$ and $\mathcal{P}_4$ have the same color set. Moreover, we have by definition that $|\mathcal{P}_1| = |(C_A - C_B) \cup C'_A| = |C_A| - |C_A \cap C_B| + |C'_A| = L - (2-\beta)L + L - \lceil \frac{2\beta L}{3} \rceil = \lfloor \frac{\beta L}{3} \rfloor$; that $|\mathcal{P}_2| = |(C_B - C_A) \cup C'_B| = |C_B| - |C_A \cap C_B| + |C'_B| = L - (2-\beta)L + L - \lceil \frac{2\beta L}{3} \rceil = \lfloor \frac{\beta L}{3} \rfloor$; and that the numbers of paths reserved for $\mathcal{P}_3$ and $\mathcal{P}_4$ are at least $|(C_A \cap C_B) - (C'_A \cup C'_B)| = (2-\beta)L - 2(L - \lceil \frac{2\beta L}{3} \rceil) \geq \lfloor \frac{\beta}{3} L \rfloor$. □

Let n and L be positive integers with $L \leq (\frac{3}{2})^n - 1$ and with $L \bmod 5 \in \{1, 3, 4\}$. It should be noted that $\lceil \frac{2}{5}L \rceil - \frac{1}{3} \geq \frac{2}{5}L + \frac{1}{15}$. Let G be a caterpillar consisting of the spine with vertices u_0, ..., u_{n+2} and edges (u_0, u_1), ..., (u_{n+1}, u_{n+2}), and of vertices v_1, ..., v_{n+1} and edges (u_1, v_1), ..., (u_{n+1}, v_{n+1}).

Lemma 12: There exists a set $\mathcal{P}'$ of directed paths with congestion L on G' such that A assigns at least $\frac{6}{5}L$ colors to the set of $2L$ paths of $\mathcal{P}'$ passing the edge (u_n, u_{n+1}), where G' is the subgraph of G induced by $\{u_0, \ldots, u_{n+1}\} \cup \{v_1, \ldots, v_n\}$.

Proof We construct $\mathcal{P}'$ according to A's coloring as follows:

1. Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the sets of L paths which start and end, respectively, at u_0.
2. Suppose that A assigns $\beta_1 L$ ($\beta_1 \geq 1$) colors to $\mathcal{P}_A \cup \mathcal{P}_B$.
3. Execute the following steps for $i = 1$ to n:
   a. If $\beta_i \geq \frac{6}{5}$, then extend the paths of $\mathcal{P}_A \cup \mathcal{P}_B$ to u_n, and quit the procedure.
   b. By using Lemma 11, choose four disjoint sets $\mathcal{P}_1 \subseteq \mathcal{P}_A$, $\mathcal{P}_2 \subseteq \mathcal{P}_B$, $\mathcal{P}_3 \subseteq \mathcal{P}_A$, and $\mathcal{P}_4 \subseteq \mathcal{P}_B$ of $\alpha_i L \equiv \lfloor \frac{\beta_i}{3}L \rfloor$ paths such that C_1, C_2, and C_3 are disjoint and that $C_3 = C_4$, where C_j is the set of colors of $\mathcal{P}_j$ for $1 \leq j \leq 4$.
   c. Extend the paths of $\mathcal{P}_1 \cup \mathcal{P}_2$ to v_i and the paths of $\mathcal{P}_3 \cup \mathcal{P}_4$ to u_{i+1}.
   d. Add sets $\mathcal{P}_5$ and $\mathcal{P}_6$ of $(1-\alpha_i)L$ paths passing from u_{i+1} to v_i and from v_i to u_{i+1}, respectively.
   e. Let $\mathcal{P}_A = \mathcal{P}_3 \cup \mathcal{P}_6$ and $\mathcal{P}_B = \mathcal{P}_4 \cup \mathcal{P}_5$.
   f. Suppose that A assigns $\beta_{i+1}L$ colors to $\mathcal{P}_A \cup \mathcal{P}_B$.

Figure 5 shows the paths $\mathcal{P}_1$, ..., $\mathcal{P}_6$ around u_i.

If there exists $1 \leq i \leq n$ such that $\beta_i \geq \frac{6}{5}$, then the lemma is immediate. Thus, we assume that $\beta_i < \frac{6}{5}$ for $1 \leq i \leq n$. Let $1 \leq i \leq n$. Since the paths of $\mathcal{P}_1 \cup \mathcal{P}_5$ intersect and so do the paths of $\mathcal{P}_4 \cup \mathcal{P}_5$, each path of $\mathcal{P}_5$ has a color either of C_2 or not of $\bigcup_{1 \leq j \leq 4} C_j$. Let $C'_2 \subseteq C_2$ be the set of colors assigned to $\mathcal{P}_5$, and let C_5 be the set of colors assigned to $\mathcal{P}_5$ but not to $\bigcup_{1 \leq j \leq 4} \mathcal{P}_j$. Similarly, let $C'_1 \subseteq C_1$ be the set of colors
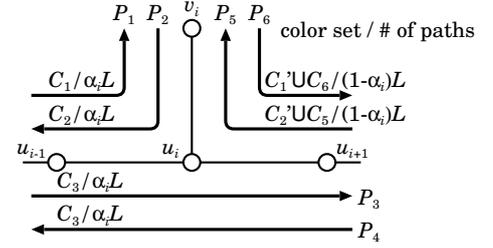

Fig. 5 Paths $\mathcal{P}_1$, ..., $\mathcal{P}_6$ around u_i for $1 \leq i \leq n$.
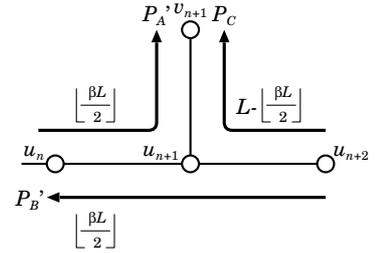

Fig. 6 Paths $\mathcal{P}'_A$, $\mathcal{P}'_B$, and $\mathcal{P}_C$ around u_{n+1}.

assigned to $\mathcal{P}_6$, and let C_6 be the set of colors assigned to $\mathcal{P}_6$ but not to $\bigcup_{1 \leq j \leq 4} \mathcal{P}_j$.

Since A uses at most $\lfloor \frac{8}{5}L \rfloor - 1$ colors, we have that $|C_1| + |C_2| + |C_3| + |C_5 \cup C_6| \leq \lfloor \frac{8}{5}L \rfloor - 1$, which yields $|C_5 \cup C_6| \leq \lfloor \frac{8}{5}L \rfloor - 3\alpha_i L - 1$. Thus, it follows that $\beta_{i+1}L = |C'_1| + |C'_2| + |C_3| + |C_5 \cup C_6| = (|C'_1| + |C_3| + |C_5 \cup C_6|) + (|\mathcal{P}_5| - |C_5|) \geq (|C'_1| + |C_3| + |C_6|) + ((1-\alpha_i)L - |C_5 \cup C_6|) \geq L + (1-\alpha_i)L - (\lfloor \frac{8}{5}L \rfloor - 3\alpha_i L - 1) = 2\lfloor \frac{\beta_i L}{3} \rfloor + \lceil \frac{2}{5}L \rceil + 1 \geq (\frac{2}{3}\beta_i + \frac{2}{5})L + \frac{1}{15}$. Therefore, we have a recurrence that $\beta_{i+1} \geq \frac{2}{3}\beta_i + \frac{2}{5} + \frac{1}{15L}$ with $\beta_1 \geq 1$, which yields $\beta_{n+1} \geq \frac{6}{5} - \frac{1}{5}(\frac{2}{3})^n + (1 - (\frac{2}{3})^n)\frac{1}{5L} \geq \frac{6}{5}$. □

Lemma 13: There exists a set $\mathcal{P}$ of directed paths with congestion L on G such that A cannot color $\mathcal{P}$ with less than $\lfloor \frac{8}{5}L \rfloor$ colors.

Proof We construct $\mathcal{P}$ according to A's coloring as follows:

1. Construct $\mathcal{P}'$ as defined in the proof of Lemma 12. Suppose that $\mathcal{P}_A$ and $\mathcal{P}_B$ are the sets of L paths passing from u_n to u_{n+1} and from u_{n+1} to u_n, respectively, and that the paths of $\mathcal{P}_A \cup \mathcal{P}_B$ have $\beta L \geq \frac{6}{5}L$ colors.
2. By using Lemma 10, choose two sets $\mathcal{P}'_A \subseteq \mathcal{P}_A$ and $\mathcal{P}'_B \subseteq \mathcal{P}_B$ of $\lfloor \frac{\beta L}{2} \rfloor$ paths such that the paths of $\mathcal{P}'_A \cup \mathcal{P}'_B$ have distinct colors.
3. Extend the paths of $\mathcal{P}'_A$ to v_{n+1} and the paths of $\mathcal{P}'_B$ to u_{n+2}.
4. Add a set $\mathcal{P}_C$ of $L - \lfloor \frac{\beta L}{2} \rfloor$ paths passing from u_{n+2} to v_{n+1}.

Figure 6 shows the paths $\mathcal{P}'_A$, $\mathcal{P}'_B$, and $\mathcal{P}_C$ around u_{n+1}. Since the paths of $\mathcal{P}'_A \cup \mathcal{P}_C$ intersect and so do the

paths of $\mathcal{P}'_B \cup \mathcal{P}_C$, no path of $\mathcal{P}_C$ has a color assigned to $\mathcal{P}'_A \cup \mathcal{P}'_B$. Thus, the number of colors necessary to color $\mathcal{P}$ is at least $|\mathcal{P}'_A| + |\mathcal{P}'_B| + |\mathcal{P}_C| = L + \lfloor \frac{\beta L}{2} \rfloor \geq \lfloor \frac{8}{5} L \rfloor$. $\square$

By Lemma 13, the proof of Theorem 3 is completed.

6. Concluding Remarks

A generalized caterpillar is a tree whose vertices of degree more than 2 are contained in a single path. Our results also hold for generalized binary caterpillars. Theorems 1 and 3 imply the complexity and the lower bound on generalized caterpillars. A path coloring algorithm on generalized binary caterpillars can be obtained by applying our algorithm to the given paths containing a vertex of the spine of a given generalized caterpillar, then coloring the uncolored paths in a simple greedy fashion. This algorithm clearly uses at most $\lceil \frac{8}{5} L \rceil$ colors, where L is the congestion of the given set of paths.

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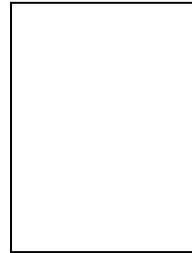
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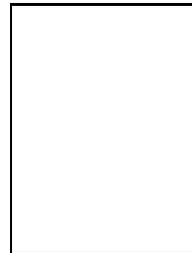
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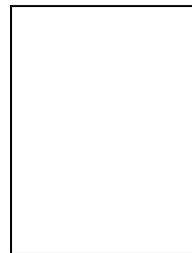
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