Random point fields for paraparticles of any order

Random Point Fields for Para-Particles of Any Order

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Abstract

Random point fields which describe gases consisting of para-particles of any order $p \in \mathrm{N}$ are given by means of the canonical ensemble approach. The analysis for the cases of the para-fermion gases are discussed in full detail and it is shown that the partition functions are p-th power of that of the usual (i.e. $p = 1$) fermion. The same is true for para-bosons.

1 Introduction

Where do the statistics of random point fields come from? We examine what kind of random point fields follow from the para-statistics of particles.

In the previous paper [9], the boson and/or fermion point fields were derived by means of the canonical ensemble approach. That is, quantum mechanical thermal systems of finite fixed number of bosons and/or fermions in the bounded box in $\mathbb{R}^d$ were considered. By taking the thermodynamic limit of the position distribution of constituents, random point fields for boson and/or fermion gases of positive finite densities and temperatures on $\mathbb{R}^d$ were obtained. There, the method was applied to construct the random point fields which describe gases consisting of para-bosons (resp. para-fermions) of order 2. In the recent proceeding article [11], the argument for para-particle gases of order 3 is developed.

In this paper, we pursue the project to the general case: we apply the method to statistical mechanics of gases which consist of para-particles of any order $p \in \mathbb{N}$. We

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will see that the random point fields obtained in this way are those of $\alpha = \pm 1/p$ given in [6]. Our main result in this paper is

Theorem *The random point field for gas of para-fermions (resp. para-bosons of low density) of order p is equal in law to the convolution of p independent copies of the usual fermion (resp. boson) point field for any $p \in \mathbb{N}$.*

We use the representation theory of the symmetric group (cf. e.g. [2, 5, 7]). Its basic facts are reviewed briefly in §2, along the line on which the quantum theory of para-particles are formulated. We state our main results in §3. Sections 4 and 5 are devoted to the full detail of the discussions on the thermodynamic limits for para-fermions and a few remarks on those for para-bosons, respectively. Some discussions are given in §6.

2 Brief review on representation of the symmetric group

We say that $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{N}^n$ is a Young frame of length n for the symmetric group $\mathcal{S}_N$ if

$$\sum_{j=1}^{n} \lambda_j = N, \quad \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0.$$

We associate the Young frame $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with the diagram of λ_1-boxes in the first row, λ_2-boxes in the second row,..., and λ_n-boxes in the n-th row. A Young tableau on a Young frame is a bijection from the numbers $1, 2, \cdots, N$ to the N boxes of the frame.

Let M_p^N be the set of all the Young frames for $\mathcal{S}_N$ which have lengths less than or equal to p. For each frame in M_p^N, let us choose one tableau from those on the frame. The choice is arbitrary but fixed. $\mathcal{T}_p^N$ denotes the set of all tableaux chosen in this way. The row stabilizer of a tableau T is denote by $\mathcal{R}(T)$, i.e., the subgroup of $\mathcal{S}_N$ consists of those elements that keep all rows of T invariant, and $\mathcal{C}(T)$ the column stabilizer whose elements preserve all columns of T.

Let us introduce the three elements

$$a(T) = \frac{1}{\#\mathcal{R}(T)} \sum_{\sigma \in \mathcal{R}(T)} \sigma, \qquad b(T) = \frac{1}{\#\mathcal{C}(T)} \sum_{\sigma \in \mathcal{C}(T)} \mathrm{sgn}(\sigma)\sigma$$

and

$$e(T) = \frac{d_T}{N!} \sum_{\sigma \in \mathcal{R}(T)} \sum_{\tau \in \mathcal{C}(T)} \mathrm{sgn}(\tau)\sigma\tau = c_T a(T) b(T)$$

of the group algebra $\mathbb{C}[\mathcal{S}_N]$ for each $T \in \mathcal{T}_p^N$, where d_T is the dimension of the irreducible representation of the symmetric group $\mathcal{S}_N$ specified by the tableau T and $c_T = d_T \times \#\mathcal{R}(T)\#\mathcal{C}(T)/N!$. As is known,

$$a(T_1)\sigma b(T_2) = b(T_2)\sigma a(T_1) = 0 \tag{2.1}$$

hold for any $\sigma \in \mathcal{S}_N$ if the frame of T_1 is larger than that of T_2 in the lexicographic order (see e.g. [5]). The relations

$$a(T)^2 = a(T), \quad b(T)^2 = b(T), \quad e(T)^2 = e(T), \quad e(T_1)e(T_2) = 0 \quad (T_1 \neq T_2) \tag{2.2}$$

also hold for $T, T_1, T_2 \in \mathcal{T}_p^N$. For later use, let us introduce

$$d(T) = e(T)a(T) = c_T a(T)b(T)a(T) \tag{2.3}$$

for $T \in \mathcal{T}_p^N$. They satisfy

$$d(T)^2 = d(T), \quad d(T_1)d(T_2) = 0 \quad (T_1 \neq T_2) \tag{2.4}$$

as is shown readily from (2.2) and (2.1). The inner product $\langle \cdot , \cdot \rangle$ of $\mathbb{C}[\mathcal{S}_N]$ is defined by

$$\langle \sigma, \tau \rangle = \delta_{\sigma\tau} \quad \text{for} \ \ \sigma, \tau \in \mathcal{S}_N$$

and the sesqui-linearity.

The left representation L and the right representation R of $\mathcal{S}_N$ on $\mathbb{C}[\mathcal{S}_N]$ are defined by

$$L(\sigma)g = L(\sigma) \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau = \sum_{\tau \in \mathcal{S}_N} g(\tau)\sigma\tau = \sum_{\tau \in \mathcal{S}_N} g(\sigma^{-1}\tau)\tau$$

and

$$R(\sigma)g = R(\sigma) \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau = \sum_{\tau \in \mathcal{S}_N} g(\tau)\tau\sigma^{-1} = \sum_{\tau \in \mathcal{S}_N} g(\tau\sigma)\tau,$$

respectively. Here and hereafter we identify $g : \mathcal{S}_N \to \mathbb{C}$ with $\sum_{\tau \in \mathcal{S}_N} g(\tau)\tau \in \mathbb{C}[\mathcal{S}_N]$. They are extended to the representation of $\mathbb{C}[\mathcal{S}_N]$ on $\mathbb{C}[\mathcal{S}_N]$ as

$$L(f)g = fg = \sum_{\sigma,\tau} f(\sigma)g(\tau)\sigma\tau = \sum_{\sigma} \big(\sum_{\tau} f(\sigma\tau^{-1})g(\tau) \big)\sigma$$

and

$$R(f)g = g\hat{f} = \sum_{\sigma,\tau} g(\sigma)f(\tau)\sigma\tau^{-1} = \sum_{\sigma} \big(\sum_{\tau} g(\sigma\tau)f(\tau) \big)\sigma,$$

where $\hat{f} = \sum_\tau \hat{f}(\tau)\tau = \sum_\tau f(\tau^{-1})\tau = \sum_\tau f(\tau)\tau^{-1}$.

The character of the irreducible representation of $\mathcal{S}_N$ corresponding to the tableau $T \in \mathcal{T}_p^N$ is obtained by

$$\chi_T(\sigma) = \sum_{\tau \in \mathcal{S}_N} \langle \tau, L(\sigma)R(e(T))\tau \rangle = \sum_{\tau \in \mathcal{S}_N} \langle \tau, \sigma\tau\widehat{e(T)} \rangle.$$

We introduce a tentative notation as in [9]

$$\chi_g(\sigma) \equiv \sum_{\tau \in \mathcal{S}_N} \langle \tau, L(\sigma)R(g)\tau \rangle = \sum_{\tau,\gamma \in \mathcal{S}_N} \langle \tau, \sigma\tau\gamma^{-1} \rangle g(\gamma) = \sum_{\tau \in \mathcal{S}_N} g(\tau^{-1}\sigma\tau) \tag{2.5}$$

for $g = \sum_\tau g(\tau)\tau \in \mathbb{C}[\mathcal{S}_N]$. Then $\chi_T = \chi_{e(T)}$ holds.

We consider representations of $\mathcal{S}_N$ on Hilbert spaces. Let $\mathcal{H}_L$ be a certain L^2 space which will be specified in the next section and $\otimes^N \mathcal{H}_L$ its N-fold Hilbert space tensor product. Let U be the representation of $\mathcal{S}_N$ on $\otimes^N \mathcal{H}_L$ defined by

$$U(\sigma)\varphi_1 \otimes \cdots \otimes \varphi_N = \varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \varphi_{\sigma^{-1}(N)} \qquad \text{for } \varphi_1, \cdots, \varphi_N \in \mathcal{H}_L,$$

or equivalently by

$$(U(\sigma)f)(x_1, \cdots, x_N) = f(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) \qquad \text{for } f \in \otimes^N \mathcal{H}_L.$$

Obviously, U is unitary: $U(\sigma)^* = U(\sigma^{-1}) = U(\sigma)^{-1}$. We extend U for $\mathbb{C}[\mathcal{S}_N]$ by linearity. Then $U(a(T))$ is an orthogonal projection because $U(a(T))^* = U(\widehat{a(T)}) = U(a(T))$ and (2.2). So are $U(b(T))$'s, $U(d(T))$'s and

$$P_{pB} = \sum_{T \in \mathcal{T}_p^N} U(d(T)). \tag{2.6}$$

Note that $\operatorname{Ran} U(d(T)) = \operatorname{Ran} U(e(T))$ because $d(T)e(T) = e(T)$ and $e(T)d(T) = d(T)$.

For para-fermions, we consider the transposed tableau T' of $T \in \mathcal{T}_p^N$ by exchanging the rows and the columns of the Young tableau T. The transpose λ' of frame λ is defined in the same way. Then T' lives on λ' if T lives on λ. Clearly

$$\mathcal{C}(T') = \mathcal{R}(T), \quad \mathcal{R}(T') = \mathcal{C}(T) \tag{2.7}$$

and we also define the projection

$$P_{pF} = \sum_{T \in \mathcal{T}_p^N} U(d(T')). \tag{2.8}$$

3 Para-statistics and random point fields

3.1 Para-fermions of order $p \in \mathbb{N}$

We first consider the quantum system of N para-fermions of order p in the box $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$. We refer the literature [3, 1, 8] for quantum mechanics of para-particles. (See also [4].) They indicate that the state space of our system is given by $\mathcal{H}_{L,N}^{pF} = P_{pF} \otimes^N \mathcal{H}_L$, where $\mathcal{H}_L = L^2(\Lambda_L)$ with Lebesgue measure is the state space of one particle system in Λ_L. We need the heat operator $G_L = e^{\beta \triangle_L}$ in Λ_L, where $\triangle_L$ is the Laplacian in Λ_L with periodic boundary conditions at $\partial\Lambda_L$. Then

$$\begin{aligned} \operatorname{spec} G_L &= \left\{ \exp(-\beta|2\pi k/L|^2) \,\middle|\, k \in \mathbb{Z}^d \right\}, \\ \frac{1}{L^d} \operatorname{Tr} G_L &= \frac{1}{L^d} \sum_{k \in \mathbb{Z}^d} \exp\left(-\beta|2\pi k/L|^2\right). \end{aligned}$$

For $k \in \mathbb{Z}^d$, $\varphi_k^{(L)}(x) = L^{-d/2} \exp(i2\pi k \cdot x/L)$ is an eigenfunction of G_L, and $\{ \varphi_k^{(L)} \}_{k \in \mathbb{Z}^d}$ forms a complete orthonormal system [CONS] of $\mathcal{H}_L$.

It is obvious that there is a CONS of $\mathcal{H}_{L,N}^{pF}$ which consists of the vectors of the form $U(d(T'))\varphi_{k_1}^{(L)} \otimes \cdots \otimes \varphi_{k_N}^{(L)}$, which are the eigenfunctions of $\otimes^N G_L$. Let $\{\Phi_k\}_{k \in \mathbb{N}}$ denote this CONS. From the canonical ensemble point of view in quantum statistical mechanics, the probability density distribution of the positions of the N free para-fermions of order p in the periodic box Λ_L at the inverse temperature β is given by

$$p_{L,N}^{pF}(x_1, \cdots, x_N) = Z_{pF}^{-1} \sum_{k \in \mathbb{N}} \overline{\Phi_k(x_1, \cdots, x_N)} \times \big((\otimes^N G_L)\Phi_k\big)(x_1, \cdots, x_N) \tag{3.1}$$

where Z_{pF} is the normalization constant. From the density (3.1), we can define the random point field of N points in Λ_L as follows (c.f. §2 of [9]). Consider the map $\Lambda_L^N \ni (x_1, \cdots, x_N) \mapsto \sum_{j=1}^N \delta_{x_j} \in Q(\mathbb{R}^d)$, where $Q(\mathbb{R}^d)$ is the space of all the point measures on $\mathbb{R}^d$. Let $\mu_{L,N}^{pF}$ be the probability measure on $Q(\mathbb{R}^d)$ induced by the map from the probability measure on Λ_L^N which has the density (3.1). By $\mathrm{E}_{L,N}^{pF}$, we denote expectation with respect to the measure $\mu_{L,N}^{pF}$. The generating functional of the point field $\mu_{L,N}^{pF}$ is given by

$$\mathrm{E}_{L,N}^{pF}\big[e^{-<f,\xi>}\big] = \int_{Q(\mathbb{R}^d)} d\mu_{L,N}^{pF}(\xi)\, e^{-<f,\xi>}$$

$$= \int_{\Lambda_L^N} \exp\big(-\sum_{j=1}^N f(x_j)\big) p_{L,N}^{pF}(x_1, \cdots, x_N)\, dx_1 \cdots dx_N$$

$$= \frac{\mathrm{Tr}_{\,\mathcal{H}_{L,N}^{pF}}\big[(\otimes^N e^{-f})(\otimes^N G_L)\big]}{\mathrm{Tr}_{\,\mathcal{H}_{L,N}^{pF}}[\otimes^N G_L]} = \frac{\sum_{T \in \mathcal{T}_p^N} \mathrm{Tr}_{\,\otimes^N \mathcal{H}_L}\big[(\otimes^N \tilde{G}_L)U(d(T'))\big]}{\sum_{T \in \mathcal{T}_p^N} \mathrm{Tr}_{\,\otimes^N \mathcal{H}_L}\big[(\otimes^N G_L)U(d(T'))\big]},$$

where f is a nonnegative continuous function on Λ_L and $\tilde{G}_L = G_L^{1/2} e^{-f} G_L^{1/2}$.

We first prove:

Lemma 1

$$E_{L,N}^{pF}\big[e^{-\langle f,\xi\rangle}\big] = \frac{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_{T'}(\sigma) \mathrm{Tr}_{\,\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L)U(\sigma)]}{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_{T'}(\sigma) \mathrm{Tr}_{\,\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]} \tag{3.2}$$

$$= \frac{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_{T'}\{\tilde{G}_L(x_i, x_j)\}_{1 \leqslant i,j \leqslant N} dx_1 \cdots dx_N}{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_{T'}\{G_L(x_i, x_j)\}_{1 \leqslant i,j \leqslant N} dx_1 \cdots dx_N}. \tag{3.3}$$

Remark 1 : The state space $\mathcal{H}_{L,N}^{pF} = P_{pF} \otimes^N \mathcal{H}_L$ is determined by $\mathcal{T}_p^N$, the choice of the tableaux T's. The different $\mathcal{T}_p^N$ gives a different subspace of $\otimes^N \mathcal{H}_L$. However, they are unitarily equivalent and the generating functional given above is not affected by the choice. In fact, $\chi_T(\sigma)$ depends only on the frame on which the tableau T is defined.

Remark 2 : $\det_{T'} A = \sum_{\sigma\in\mathcal{S}_N} \chi_{T'}(\sigma)\prod_{i=1}^N A_{i\sigma(i)}$ in (3.3) is known as immanant.
Proof : These expressions are derived by the relations

$$\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big[(\otimes^N G_L)U(d(T'))\big] = \mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big[(\otimes^N G_L)U(e(T'))\big]$$

and

$$\sum_{\sigma\in\mathcal{S}_N}\chi_g(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big[(\otimes^N G_L)U(\sigma)\big] = N!\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big[(\otimes^N G_L)U(g)\big], \tag{3.4}$$

with $g = e(T')$. These relations can be shown by the use of (2.5), the cyclic property of the trace and the commutativity of $U(\tau)$ with $\otimes^N G_L$. For details, see [9]. □

Now, let us consider the thermodynamic limit

$$L,\ N\to\infty, \quad \rho_L\equiv N/L^d\to\rho>0. \tag{3.5}$$

In the following, f is a nonnegative continuous function on $\mathbb{R}^d$ which has a compact support, and is fixed through the thermodynamic limit $\Lambda_L \nearrow \mathbb{R}^d$. We identify the restriction of f to Λ_L as f in Lemma 1. We get the limiting random point field on $\mathbb{R}^d$.

Theorem 2 *The finite random point fields for para-fermions of order p defined above converge weakly to the point field whose generating functional is given by*

$$\mathrm{E}_\rho^{pF}\big[e^{-\langle f,\xi\rangle}\big] = \mathrm{Det}\big[1-\sqrt{1-e^{-f}}r_*G(1+r_*G)^{-1}\sqrt{1-e^{-f}}\big]^p$$

in the thermodynamic limit (3.5), where $r_*\in(0,\infty)$ *is determined by*

$$\frac{\rho}{p} = \int\frac{dp}{(2\pi)^d}\frac{r_*e^{-\beta|p|^2}}{1+r_*e^{-\beta|p|^2}} = (r_*G(1+r_*G)^{-1})(x,x),$$

$G = e^{\beta\triangle}$ *is the heat operator on the whole space* $\mathbb{R}^d$ *and* Det *stands for the Fredholm determinant.*

3.2 Para-bosons of order $p\in\mathbb{N}$

We next consider the quantum system of N para-bosons of order p in the box Λ_L. The state space of the system is given by $\mathcal{H}_{L,N}^{pB} = P_{pB}\otimes^N\mathcal{H}_L$. As for the para-fermions' case in the previous subsection, the point field of N free para-bosons of order p can be defined. Its generating functional is given by

$$E_{L,N}^{pB}\big[e^{-\langle f,\xi\rangle}\big] = \frac{\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N\tilde{G}_L)P_{pB}]}{\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)P_{pB}]},$$

where f is a nonnegative continuous function on Λ_L. Then, we have:

Lemma 3

$$E_{L,N}^{pB}\big[e^{-\langle f,\xi\rangle}\big] = \frac{\sum_{T\in\mathcal{T}_p^N}\sum_{\sigma\in\mathcal{S}_N}\chi_T(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N\tilde{G}_L)U(\sigma)]}{\sum_{T\in\mathcal{T}_p^N}\sum_{\sigma\in\mathcal{S}_N}\chi_T(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]} \tag{3.6}$$

$$= \frac{\sum_{T\in\mathcal{T}_p^N}\int_{\Lambda_L^N}\det_T\{\tilde{G}_L(x_i,x_j)\}dx_1\cdots dx_N}{\sum_{T\in\mathcal{T}_p^N}\int_{\Lambda_L^N}\det_T\{G_L(x_i,x_j)\}dx_1\cdots dx_N}. \tag{3.7}$$

We again consider the thermodynamic limit (3.5). We get the limiting random point field on $\mathbb{R}^d$ for the low density region:

Theorem 4 *The finite random point fields for para-bosons of order p defined above converge weakly to the random point field whose Laplace transform is given by*

$$\mathrm{E}_\rho^{pB}\big[e^{-\langle f,\xi\rangle}\big] = \mathrm{Det}\big[1+\sqrt{1-e^{-f}}\,r_*G(1-r_*G)^{-1}\sqrt{1-e^{-f}}\big]^{-p}$$

in the thermodynamic limit, where $r_* \in (0,1)$ is determined by

$$\frac{\rho}{p} = \int \frac{dp}{(2\pi)^d}\frac{r_*e^{-\beta|p|^2}}{1-r_*e^{-\beta|p|^2}} = (r_*G(1-r_*G)^{-1})(x,x),$$

if

$$\frac{\rho}{p} < \rho_c \equiv \int_{\mathbb{R}^d}\frac{dp}{(2\pi)^d}\frac{e^{-\beta|p|^2}}{1-e^{-\beta|p|^2}}.$$

Remark : The high density region $\rho \geqslant p\rho_c$ is related to the Bose-Einstein condensation. We need a different analysis for the region. See [10] for the case of $p = 1$ and 2.

4 Proof of theorem 2

It is enough to show the convergence of the generating functionals. In the rest of this paper, we use the results in [9] frequently. We refer them as, e.g., Lemma I.3.2 for Lemma 3.2 of [9]. Although those in [9] are results for $p = 1$, their arguments hold for general $p \in \mathbb{N}$ with obvious changes. Let ψ_T be the character of the induced representation $\mathrm{Ind}_{\mathcal{R}(T)}^{\mathcal{S}_N}[\mathbf{1}]$, where $\mathbf{1}$ is the one dimensional representation $\mathcal{R}(T) \ni \sigma \to 1$, i.e.,

$$\psi_T(\sigma) = \sum_{\tau\in\mathcal{S}_N}\langle\tau, L(\sigma)R(a(T))\tau\rangle = \chi_{a(T)}(\sigma).$$

Since the characters χ_T and ψ_T depend only on the frame on which the tableau T lives, not on T itself, we also use the notation χ_λ and ψ_λ ($\lambda \in M_p^N$) instead of χ_T and ψ_T, respectively.

Let δ be the frame $(p-1,\cdots,2,1,0)$. Generalize ψ_μ to those $\mu = (\mu_1,\cdots,\mu_p) \in \mathbb{Z}^p$ which satisfies $\sum_{j=1}^p \mu_j = N$ by

$$\psi_\mu = 0 \qquad \text{for} \ \ \mu \in \mathbb{Z}^p - \mathbb{Z}_+^p$$

and

$$\psi_\mu = \psi_{\pi\mu} \quad \text{for} \quad \mu \in \mathbb{Z}_+^p \quad \text{and} \quad \pi \in \mathcal{S}_p \quad \text{such that} \quad \pi\mu \in M_p^N,$$

where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$. Then the determinantal form [2] can be written as

$$\chi_\lambda = \sum_{\pi\in\mathcal{S}_p}\mathrm{sgn}\pi\,\psi_{\lambda+\delta-\pi\delta}. \tag{4.1}$$

Let us recall the relations

$$\chi_{T'}(\sigma) = \operatorname{sgn}\sigma\,\chi_T(\sigma), \qquad \varphi_{T'}(\sigma) = \operatorname{sgn}\sigma\,\psi_T(\sigma),$$

where

$$\varphi_{T'}(\sigma) = \sum_\tau \langle \tau, L(\sigma)R(b(T'))\tau\rangle = \chi_{b(T')}(\sigma)$$

denotes the character of the induced representation $\mathrm{Ind}_{\mathcal{C}(T')}^{\mathcal{S}_N}[\,\mathrm{sgn}\,]$, where sgn is the representation $\mathcal{C}(T') = \mathcal{R}(T) \ni \sigma \mapsto \operatorname{sgn}\sigma$. Then we have a variant of (4.1):

$$\chi_{\lambda'} = \sum_{\pi\in\mathcal{S}_p} \mathrm{sgn}\pi\,\varphi_{\lambda'+\delta'-(\pi\delta)'}. \tag{4.2}$$

Now let us consider the denominator of (3.2). Let $T \in \mathcal{T}_p^N$ live on $\mu = (\mu_1, \cdots, \mu_p) \in M_p^N$. Thanks to (3.4) for $g = b(T')$, we have

$$\sum_{\sigma\in\mathcal{S}_N} \varphi_{T'}(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big[(\otimes^N G_L)U(\sigma)\big] = N!\mathrm{Tr}_{\otimes^N\mathcal{H}_L}\big[(\otimes^N G_L)U(b(T'))\big]$$

$$= N!\prod_{j=1}^p \mathrm{Tr}_{\otimes^{\mu_j}\mathcal{H}_L}\big[(\otimes^{\mu_j}G_L)A_{\mu_j}\big],$$

where $A_n = \sum_{\tau\in\mathcal{S}_n}\mathrm{sgn}(\tau)U(\tau)/n!$ is the anti-symmetrization operator on $\otimes^n\mathcal{H}_L$. In the last step, we have used

$$b(T') = \prod_{j=1}^p \sum_{\sigma\in\mathcal{R}_j} \frac{\mathrm{sgn}\sigma}{\#\mathcal{R}_j}\sigma,$$

where $\mathcal{R}_j$ is the symmetric group of μ_j numbers which lie on the j-th row of the tableau T. Now (4.2) yields

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{S}_N} \chi_{\lambda'}(\sigma)\mathrm{Tr}_{\,\otimes^N\mathcal{H}_L}\big[(\otimes^N G_L)U(\sigma)\big] \\
&= \sum_{\pi\in\mathcal{S}_p}\mathrm{sgn}\,\pi \sum_{\sigma\in\mathcal{S}_N}\varphi_{\lambda'+\delta'-(\pi\delta)'}(\sigma)\mathrm{Tr}_{\,\otimes^N\mathcal{H}_L}\big[(\otimes^N G_L)U(\sigma)\big] \\
&= N!\sum_{\pi\in\mathcal{S}_p}\mathrm{sgn}\,\pi\prod_{j=1}^p \mathrm{Tr}_{\,\otimes^{\lambda_j-j+\pi(j)}\mathcal{H}_L}\big[(\otimes^{\lambda_j-j+\pi(j)}G_L)A_{\lambda_j-j+\pi(j)}\big].
\end{aligned} \tag{4.3}$$

Here we understand that $\mathrm{Tr}_{\,\otimes^n\mathcal{H}_L}\big((\otimes^n G_L)A_n\big) = 1$ if $n = 0$ and $= 0$ if $n < 0$ in the last expression. Applying the Cauchy integral formula to

$$\mathrm{Det}[1+zJ] = \sum_{n=0}^\infty z^n \mathrm{Tr}_{\,\otimes^n\mathcal{H}}[(\otimes^n J)A_n]$$

where J is a trace class operator, we obtain that

$$\mathrm{Tr}_{\,\otimes^n\mathcal{H}}[(\otimes^n G_L)A_n] = \oint_{S_r(0)} \frac{dz}{2\pi i z^{n+1}}\mathrm{Det}[1+zG_L], \tag{4.4}$$

where $S_r(\zeta) = \{z \in \mathbb{C}; |z-\zeta| = r\}$. Note that $r > 0$ can be chosen arbitrary and the right hand side equals 1 for $n = 0$ and 0 for $n < 0$. Then we have the following expression of the denominator of (3.2):

$$\begin{aligned}
&\sum_{\lambda \in \mathcal{M}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_{\lambda'}(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\sigma)] \\
&= N! \sum_{\lambda \in \mathcal{M}_p^N} \sum_{\pi \in \mathcal{S}_p} \mathrm{sgn}\pi \oint \cdots \oint_{S_r(0)^p} \prod_{j=1}^{p} \frac{\mathrm{Det}(1+z_j G_L)\, dz_j}{2\pi i z_j^{\lambda_j - j + \pi(j) + 1}}. \\
&= N! \sum_{\lambda \in \mathcal{M}_p^N} \oint \cdots \oint_{S_r(0)^p} \frac{\Delta_p(z_1, \cdots, z_p) \big[\prod_{j=1}^p \mathrm{Det}(1+z_j G_L) dz_j \big]}{\prod_{j=1}^p 2\pi i z_j^{\lambda_j + p - j + 1}}, \qquad (4.5)
\end{aligned}$$

where $\Delta_p(z_1, \cdots, z_p)$ is the Vandermondian given by

$$\Delta_p(z_1, \cdots, z_p) \equiv \prod_{1 \leqslant i < j \leqslant p} (z_i - z_j) = \begin{vmatrix} z_1^{p-1} & z_2^{p-1} & \cdots & z_p^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & \cdots & z_p \\ 1 & 1 & \cdots & 1 \end{vmatrix}. \qquad (4.6)$$

In the following, we simply write $\Delta_p(\{z\})$ for $\Delta(z_1, \cdots, z_p)$ when there is no danger of confusion.

To make the thermodynamic limit procedure explicit, we take a sequence $\{L_N\}_{N\in\mathbb{N}}$ which satisfies $N/L_N^d \to \rho$ as $N \to \infty$. In the following, we set $r = r_N \in [0, \infty)$ to be the unique solution of

$$\mathrm{Tr}\, \frac{rG_{L_N}}{1 + rG_{L_N}} = k \qquad (4.7)$$

where

$$k = \Big\lfloor \frac{N}{p} + \frac{p-1}{2} \Big\rfloor \qquad (4.8)$$

is the averaged length of the rows in the Young tableau and $\lfloor \cdot \rfloor$ represents the integer part. The existence and the uniqueness of the solution follow from the fact that the left-hand side of (4.7) is a continuous and monotone function of r. For details, see Lemma I.3.2 [for $\alpha = -1$].

We also put

$$v_N = \mathrm{Tr}\, \frac{r_N G_{L_N}}{(1 + r_N G_{L_N})^2}. \qquad (4.9)$$

We will suppress the N dependence of r_N, v_N and L_N. Since $r_N \to r_*$ in the thermodynamic limit, we have $k/(2+r_*) \leqslant v_N \leqslant k$ for large enough N. See Lemma I.3.5. [There r_N and r_* are written as z_N and z_* respectively.]

Put

$$(4.5) = \frac{N!\, \mathrm{Det}[1 + rG_L]^p}{(\sqrt{2\pi})^p (\sqrt{v})^{1+p(p-1)/2} r^N} J_p. \qquad (4.10)$$

Then we have:

Lemma 5

$$\lim_{N\to\infty} J_p = \frac{1}{p!}\int_{\mathbb{R}^p} |\Delta_p(y_1,\cdots,y_p)|\,\delta\Big(\sum_{j=1}^p y_j\Big)\prod_{j=1}^p e^{-y_j^2/2}dy_j > 0. \tag{4.11}$$

Proof : We set

$$\nu_j = \lambda_j + p - j - k.$$

Then we have

$$\nu = (\nu_1,\cdots,\nu_p)\in\mathbb{Z}^p, \tag{4.12a}$$

$$\sum_{j=1}^p \nu_j = \nu_0 \equiv N + \frac{p(p-1)}{2} - pk \in [0,p), \tag{4.12b}$$

$$\nu_1 > \nu_2 > \cdots > \nu_p \geqslant -k. \tag{4.12c}$$

The parametrization

$$z_j = r\exp(ix_j/\sqrt{v}) \qquad (j=1,\cdots,p)$$

yields

$$\mathrm{Det}[1+z_jG_L] = \mathrm{Det}[1+rG_L]\mathrm{Det}\big[1+(z_j-r)G_L(1+rG_L)^{-1}\big], \tag{4.13a}$$

$$z_j - r = r(e^{ix_j/\sqrt{v}}-1) = r\big(i\sin(x_j/\sqrt{v}) - 2\sin^2(x_j/2\sqrt{v})\big), \tag{4.13b}$$

$$dz_j = ire^{ix_j/\sqrt{v}}dx_j/\sqrt{v}, \tag{4.13c}$$

$$\Delta_p(\{z\}) = r^{1+2+\cdots+(p-1)}\Delta_p(\{e^{ix/\sqrt{v}}\}) \tag{4.13d}$$

and

$$J_p = \sum_\nu \Big(\prod_{j=1}^p\int_{-\pi\sqrt{v}}^{\pi\sqrt{v}}\frac{dx_j}{\sqrt{2\pi}}\Big)\Delta_p(\{e^{ix/\sqrt{v}}\})(\sqrt{v})^{1-p+p(p-1)/2} \times\Big(\prod_{j=1}^p e^{-i(\nu_j+k)x_j/\sqrt{v}}\mathrm{Det}\Big[1+(e^{ix_j/\sqrt{v}}-1)\frac{rG_L}{1+rG_L}\Big]\Big), \tag{4.14}$$

where the summation on ν is taken over all ν satisfying (4.12).

We consider two regions of $x\in(-\pi\sqrt{v},\pi\sqrt{v}]$

1. small x region: $|x|\leqslant v^{1/12}$,
2. large x region: $|x| > v^{1/12}$.

In the large x region, we have

$$\begin{aligned}&\Big|\mathrm{Det}\Big[1+(z-r)\frac{G_L}{1+rG_L}\Big]\Big|^2\\ &= \mathrm{Det}\Big[1-4\sin^2\big(\frac{x}{2\sqrt{v}}\big)\frac{rG_L}{1+rG_L}\Big(1-\frac{rG_L}{1+rG_L}\Big)\Big]\\ &\leqslant \mathrm{Det}\Big[1-\frac{4}{1+r}\sin^2\big(\frac{x}{2\sqrt{v}}\big)\frac{rG_L}{1+rG_L}\Big]\\ &\leqslant \exp\Big(-\frac{4}{1+r}\sin^2\big(\frac{x}{2\sqrt{v}}\big)\mathrm{Tr}\,\frac{rG_L}{1+rG_L}\Big)\leqslant\exp(-\mathrm{const}\,N^{1/6}),\end{aligned}$$

using $0 \leqslant G_L \leqslant 1$ and $v_N = O(k) = O(N)$ and the boundedness of $r = r_N > 0$ uniformly in N.

In the small x region, we have the convergent expansion

$$\begin{aligned}
&\mathrm{Det}\Big[1 + (z-r)\frac{G_L}{1+rG_L}\Big] \\
&= \exp\Bigg[(e^{ix/\sqrt{v}}-1)k - \frac{1}{2}(e^{ix/\sqrt{v}}-1)^2(k-v) \\
&\qquad + \sum_{\ell=3}^{\infty}\frac{(-1)^{\ell-1}}{\ell}(e^{ix/\sqrt{v}}-1)^{\ell}\mathrm{Tr}\left(\frac{rG_L}{1+rG_L}\right)^{\ell}\Bigg] \\
&= \big(1 + \sum_{\ell=3}^{n-1} c_\ell x^\ell + R_n(x)\big)\exp\Big(\frac{ikx}{\sqrt{v}} - \frac{1}{2}x^2\Big),
\end{aligned} \tag{4.15}$$

where

$$|c_\ell| \;\leq\; \mathrm{const}\, N^{-\ell/6}, \qquad (\,\ell = 3, \cdots, n-1\,) \tag{4.16}$$

and

$$||R_n||_\infty \equiv \sup_{|x|\leqslant v^{1/12}} |R_n(x)| = O\big(N^{-n/12}\big)$$

hold. We put

$$\sum_{\ell=3}^{n-1} c_\ell x^\ell = \gamma(x).$$

We choose n in (4.15) so large that

$$\sum_{\nu}(\sqrt{v})^{1-p+p(p-1)/2}||\Delta_p||_\infty ||R_n||_\infty = o(1) \tag{4.17}$$

holds, i.e., $n > 3(p-1)(p+2)$.

These arguments show that it is enough to consider the contribution from the small x region, and we have

$$\begin{aligned}
J_p &= \Bigg\{\sum_{\nu}\Bigg(\prod_{j=1}^{p}\int_{-v^{1/12}}^{v^{1/12}}\frac{dx_j}{\sqrt{2\pi}}\,e^{-i\nu_j x_j/\sqrt{v}-x_j^2/2}(1+\gamma(x_j))\Bigg) \\
&\qquad \times\Delta_p(\{e^{ix/\sqrt{v}}\})(\sqrt{v})^{1-p+p(p-1)/2}\Bigg\} + o(1), \\
&= \Bigg\{\sum_{\nu}\Bigg(\prod_{j=1}^{p}\Big(1+\gamma\Big(i\sqrt{v}\frac{\partial}{\partial\nu_j}\Big)\Big)\int_{-\infty}^{\infty}\frac{dx_j}{\sqrt{2\pi}}\,e^{-i\nu_j x_j/\sqrt{v}-x_j^2/2}\Bigg) \\
&\qquad \times\Delta_p(\{e^{ix/\sqrt{v}}\})(\sqrt{v})^{1-p+p(p-1)/2}\Bigg\} + o(1).
\end{aligned} \tag{4.18}$$

Thanks to the multi-linearity of the determinant Δ_p, we have

$$\begin{aligned}
&\Big(\prod_{j=1}^{p}\int_{-\infty}^{\infty}\frac{dx_j}{\sqrt{2\pi}}\,e^{-i\nu_j x_j/\sqrt{v}-x_j^2/2}\Big)\Delta_p(\{e^{ix/\sqrt{v}}\})\\
&= \det\Big\{\int_{-\infty}^{\infty}\frac{dx_j}{\sqrt{2\pi}}e^{i(\ell-\nu_j)x_j/\sqrt{v}-x_j^2/2}\Big\}_{\substack{p-1\geqslant\ell\geqslant 0\\ 1\leqslant j\leqslant p}}\\
&= \det\Big\{e^{-(\ell-\nu_j)^2/2v}\Big\}_{\substack{p-1\geqslant\ell\geqslant 0\\ 1\leqslant j\leqslant p}}\\
&= \Delta_p\big(\{e^{\nu/v}\}\big)\exp\Big(-\sum_{j=1}^{p}\frac{\nu_j^2}{2v}-\sum_{\ell=0}^{p-1}\frac{\ell^2}{2v}\Big). \qquad (4.19)
\end{aligned}$$

Since $(\sqrt{v})^{p(p-1)/2}\Delta_p(\{e^{\nu/v}\})=\Delta_p(\{\sqrt{v}(e^{\nu/v}-1)\})$, the summation over all ν satisfying the condition (4.12) yields

$$\begin{aligned}
&\lim_{N\to\infty}\sum_{\nu}(\sqrt{v})^{1-p+p(p-1)/2}\times(4.19)\\
&=\int_{y_1>\cdots>y_p}\delta(\sum_{j=1}^{p}y_j)\Delta_p(y_1,\cdots,y_p)\prod_{j=1}^{p}e^{-y_j^2/2}dy_j. \qquad (4.20)
\end{aligned}$$

Here we have put $y_j=\nu_j/\sqrt{v}$ and regarded $(\sqrt{v})^{1-p}\sum_\nu$ as the integral of the suitable step function. Then, $(\sqrt{v})^{1-p}\sum_\nu\to\int dy\,\delta(\sum_j y_j)$ is derived by the use of the dominated convergence theorem. The limit of the main term of (4.18) is given by (4.20), which is equal to (4.11). We may see that the limit of the remainder vanishes from (4.16) and the convergence of

$$\begin{aligned}
&\sum_{\nu}\Big(\prod_{j=1}^{p}(\sqrt{v})^{a_j}\frac{\partial^{a_j}}{\partial\nu_j^{a_j}}\int_{-\infty}^{\infty}\frac{dx_j}{\sqrt{2\pi}}\,e^{-i\nu_j x_j/\sqrt{v}-x_j^2/2}\Big)\\
&\qquad\times\Delta_p(\{e^{ix/\sqrt{v}}\})(\sqrt{v})^{1-p+p(p-1)/2}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\nu}(\sqrt{v})^{1-p}\Big(\prod_{j=1}^{p}(\sqrt{v})^{a_j}\frac{\partial^{a_j}}{\partial\nu_j^{a_j}}\Big)\Delta_p\big(\{\sqrt{v}(e^{\nu/v}-1)\}\big)\exp\Big(-\sum_{j=1}^{p}\frac{\nu_j^2}{2v}-\sum_{\ell=0}^{p-1}\frac{\ell^2}{2v}\Big)\\
&\longrightarrow \int_{y_1>\cdots>y_p}\delta(\sum_{j=1}^{p}y_j)\Big(\prod_{j=1}^{p}\frac{\partial^{a_j}}{\partial y_j^{a_j}}\Big)\Delta_p(y_1,\cdots,y_p)e^{-\sum_{j=1}^{p}y_j^2/2}\Big(\prod_{j=1}^{p}dy_j\Big). \qquad (4.21)
\end{aligned}$$

We obtain this convergence by performing the differentiations in the second and the third members of (4.21) and applying the dominated convergence theorem. □

The numerator is obtained just in the same way. That is, we replace G_L by $\tilde{G}_L = G_L^{1/2}e^{-f}G_L^{1/2}$ and introduce $\tilde{r}=\tilde{r}_N$ and $\tilde{v}=\tilde{v}_N$ by

$$\mathrm{Tr}\,\frac{\tilde{r}_N\tilde{G}_{L_N}}{1+\tilde{r}_N\tilde{G}_{L_N}}=k,\qquad \mathrm{Tr}\,\frac{\tilde{r}_N\tilde{G}_{L_N}}{(1+\tilde{r}_N\tilde{G}_{L_N})^2}=\tilde{v}_N.$$

Since $G_L - \tilde{G}_L$ is a positive operator of trace class such that $\mathrm{Tr}\,(G_L - \tilde{G}_L) = O(||1-e^{-f})||_1)$ and $\max\{\mathrm{spec}\,G_L\} - \max\{\mathrm{spec}\,\tilde{G}_L\} = O(L^{-d})$, it follows from the definitions of $r_N, \tilde{r}_N, v_N$ and $\tilde{v}_N$ that

$$0 \leqslant \tilde{r}_N - r_N = O(N^{-1}), \quad |\tilde{v}_N - v_N| = O(1). \tag{4.22}$$

See Lemma I.3.5 and Lemma I.3.6 for details.

We define $\tilde{J}_p$ similarly ($\tilde{v}_N$ is used instead of v_N) and we get

$$\lim_{N\to\infty} \tilde{J}_p = \frac{1}{p!}\int_{\mathbb{R}^p} |\Delta_p(y_1,\cdots,y_p)|\,\delta\Big(\sum_{j=1}^p y_j\Big)\prod_{j=1}^p e^{-y_j^2/2}dy_j > 0$$

by the very same argument as in the proof of Lemma 5.

Thus we have (writing $r = r_N$, $\tilde{r} = \tilde{r}_N$ and $L = L_N$)

$$\begin{aligned}
(3.2) \quad &= \quad \left(\frac{\mathrm{Det}[1+\tilde{r}\tilde{G}_L]}{\mathrm{Det}[1+rG_L]}\right)^p \left(\frac{r}{\tilde{r}}\right)^N \left(\frac{v}{\tilde{v}}\right)^{p(p-1)/4+1/2}\frac{\tilde{J}_p}{J_p} \\
&= \quad \left(\frac{\mathrm{Det}[1+r\tilde{G}_L]}{\mathrm{Det}[1+rG_L]}\right)^p \mathrm{Det}\Big[1+\frac{(r-\tilde{r})}{1+\tilde{r}\tilde{G}_L}\tilde{G}_L\Big]^{-p} \\
&\quad \times\Big(1+\frac{r-\tilde{r}}{\tilde{r}}\Big)^N\left(\frac{v}{\tilde{v}}\right)^{p(p-1)/4+1/2}\frac{\tilde{J}_p}{J_p}.
\end{aligned} \tag{4.23}$$

Here

$$\begin{aligned}
\left(\frac{\mathrm{Det}[1+r\tilde{G}_L]}{\mathrm{Det}[1+rG_L]}\right)^p &= \mathrm{Det}\Big[1+\frac{r}{1+rG_L}(\tilde{G}_L - G_L)\Big]^p \\
&= \mathrm{Det}\Big[1-\sqrt{1-e^{-f}}\frac{rG_L}{1+rG_L}\sqrt{1-e^{-f}}\Big]^p \\
&\to \mathrm{Det}\Big[1-\sqrt{1-e^{-f}}\frac{r_*G}{1+r_*G}\sqrt{1-e^{-f}}\Big]^p
\end{aligned}$$

holds. For details, we refer Proposition I.3.9 (and the argument on (c) in the proof of Theorem I.3.1). The remaining factor of (4.23) tends to 1 as $N \to \infty$ since $v/\tilde{v} \to 1$, $\tilde{J}_p/J_p \to 1$ and

$$\begin{aligned}
& N\log\Big(1+\frac{r-\tilde{r}}{\tilde{r}}\Big) - p\log\mathrm{Det}\Big[1+\frac{(r-\tilde{r})}{1+\tilde{r}\tilde{G}_L}\tilde{G}_L\Big] \\
= \quad & N\frac{r-\tilde{r}}{\tilde{r}} - p\frac{r-\tilde{r}}{\tilde{r}}\mathrm{Tr}\,\frac{\tilde{r}\tilde{G}_L}{1+\tilde{r}\tilde{G}_L} + O(N^{-1}) \\
= \quad & \frac{r-\tilde{r}}{\tilde{r}}(N-pk) + O(N^{-1}) = O(N^{-1}).
\end{aligned}$$

Finally, we also get

$$\frac{\rho}{p} = \lim_{N\to\infty}\frac{1}{L_N^d}\mathrm{Tr}\,\frac{r_N G_{L_N}}{1+r_N G_{L_N}} = (r_*G(1+r_*G)^{-1})(x,x)$$

from (4.7), (4.8), $N/L_N^d \to \rho$ and Proposition I.3.9. This completes the proof of Theorem 2. □

5 Proof of theorem 4

In the case of para bosons, we use the formula (4.1) instead of (4.2) and the following formula

$$\mathrm{Tr}_{\otimes^n\mathcal{H}}[(\otimes^n G_L)S_n] = \oint_{S_r(0)} \frac{dz}{2\pi i z^{n+1}} \mathrm{Det}[1-zG_L]^{-1},$$

which is derived from the generalized Vere-Jones' formula [12, 6, 9] as in (4.4). Here $S_n = \sum_{\tau\in\mathcal{S}_n} U(\tau)/n!$ is the symmetrization operator on $\otimes^n\mathcal{H}_L$ and $r \in (0,1)$ in this case. Then, we get the following expression of the denominator of (3.6):

$$\begin{aligned}
&\sum_{\lambda\in\mathcal{M}_p^N}\sum_{\sigma\in\mathcal{S}_N} \chi_\lambda(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\sigma)] \\
=& \sum_{\lambda\in\mathcal{M}_p^N}\sum_{\sigma\in\mathcal{S}_N}\sum_{\pi\in\mathcal{S}_p} \mathrm{sgn}\pi\, \psi_{\lambda+\delta-\pi\delta}(\sigma)\mathrm{Tr}_{\otimes^N\mathcal{H}_L}[(\otimes^N G_L)U(\sigma)] \\
=& N!\sum_{\lambda\in\mathcal{M}_p^N}\sum_{\pi\in\mathcal{S}_p} \mathrm{sgn}\pi \prod_{j=1}^{p} \mathrm{Tr}_{\otimes^{\lambda_j-j+\pi(j)}\mathcal{H}_L}[(\otimes^{\lambda_j-j+\pi(j)}G_L)S_{\mu_j}] \\
=& N!\sum_{\lambda\in\mathcal{M}_p^N}\sum_{\pi\in\mathcal{S}_p} \mathrm{sgn}\pi \oint\cdots\oint_{S_r(0)^p}\prod_{j=1}^{p}\frac{dz_j}{2\pi i z_j^{\lambda_j-j+\pi(j)+1}\mathrm{Det}[1-z_jG_L]} \\
=& N!\sum_{\lambda\in\mathcal{M}_p^N}\oint\cdots\oint_{S_r(0)^p}\frac{\Delta_p(z_1,\cdots,z_p)\,dz_1\cdots dz_p}{\left(\prod_{j=1}^{p}2\pi i z_j^{\lambda_j+p-j+1}\mathrm{Det}[1-z_jG_L]\right)},
\end{aligned} \tag{5.1}$$

where $\Delta_p(z_1,\cdots,z_p)$ is the Vandermondian introduced in the previous section.

We choose a sequence $\{L_N\}_{N\in\mathbb{N}}$ which satisfies $N/L_N^d \to \rho$ as $N\to\infty$. In this case, $r_N \in (0,1)$ denotes the unique solution of

$$\mathrm{Tr}\,\frac{rG_{L_N}}{1-rG_{L_N}} = k \tag{5.2}$$

where

$$k = \lfloor \frac{N}{p} + \frac{p-1}{2} \rfloor \tag{5.3}$$

as in (4.8). We put

$$v_N = \mathrm{Tr}\Big[\frac{r_N G_{L_N}}{(1-r_N G_{L_N})^2}\Big]. \tag{5.4}$$

The remaining parts are almost the same as those in the para-fermion case. The reader may complete the proof of Theorem 4, following the previous arguments with the obvious changes.

6 Discussion

We have shown that

the generating functional of the gas of para-fermions (resp. para-bosons of low density) of order p is equal to the p-th power of the generating functional of fermion (resp. boson) gas.

The random point fields which we have obtained in this paper are a subset of those in [6], where various properties of the point fields are examined. On the other hand, the authors of [6] obtained the point fields which do not follow from the representation theory of the symmetric groups which we discussed in this paper. Therefore it is interesting to consider physical interpretations of these point fields. See e.g. [13].

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