Weak-type (1, 1) estimates for parabolic singular integrals

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WEAK-TYPE $(1, 1)$ ESTIMATES FOR PARABOLIC SINGULAR INTEGRALS

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Abstract We prove weak-type $(1, 1)$ estimates for rough parabolic singular integrals on $\mathbb{R}^2$ under the $L \log L$ condition on their kernels.

Keywords: parabolic singular integrals; weak-type $(1, 1)$ estimates; rough operators

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1. Introduction

Let $\{A_t\}_{t>0}$ be a dilation group on $\mathbb{R}^n$ defined by $A_t = t^P = \exp((\log t)P)$, where P is an $n \times n$ real matrix whose eigenvalues have positive real parts. We assume $n \geqslant 2$. There is a non-negative function r on $\mathbb{R}^n$ satisfying $r(A_t x) = tr(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. We may assume the following:

(i) the function r is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$;

(ii) $r(x + y) \leqslant C_0(r(x) + r(y))$ for some $C_0 \geqslant 1$, $r(x) = r(-x)$;

(iii) if $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$, then $\Sigma = \{\theta \in \mathbb{R}^n : \langle B\theta, \theta\rangle = 1\}$ for a positive symmetric matrix B, where $\langle \cdot\,, \cdot\rangle$ denotes the inner product in $\mathbb{R}^n$;

(iv) we have $\mathrm{d}x = t^{\gamma-1}\,\mathrm{d}\sigma\,\mathrm{d}t$, that is,

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = \int_0^\infty \int_\Sigma f(A_t\theta)t^{\gamma-1}\,\mathrm{d}\sigma(\theta)\,\mathrm{d}t$$

for appropriate functions f, where $\mathrm{d}\sigma$ is a C^∞ measure on Σ and $\gamma = \operatorname{tr} P$;

(v) there are positive constants c_1, c_2, c_3, c_4, α_1, α_2, β_1 and β_2 such that

$$\begin{aligned} c_1|x|^{\alpha_1} \leqslant r(x) \leqslant c_2|x|^{\alpha_2} \quad &\text{if } r(x) \geqslant 1, \\ c_3|x|^{\beta_1} \leqslant r(x) \leqslant c_4|x|^{\beta_2} \quad &\text{if } r(x) \leqslant 1. \end{aligned}$$

(See [**2**, **9**, **14**] for more details.)

Let K be a locally integrable function on $\mathbb{R}^n \setminus \{0\}$ satisfying

$$K(A_t x) = t^{-\gamma} K(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\};$$

and

$$\int_{a<r(x)<b} K(x)\,\mathrm{d}x = 0 \quad \text{for all } a, b \text{ with } a < b.$$

Define

$$Tf(x) = \text{p.v.} \int f(y) K(x-y)\,\mathrm{d}y.$$

Let

$$D_0 = \{x \in \mathbb{R}^n : 1 \leqslant r(x) \leqslant 2\} \quad \text{and} \quad K_0(x) = K(x)\chi_{D_0}(x), \tag{1.1}$$

where χ_S is the characteristic function of a set S. If $K_0 \in L \log L(\mathbb{R}^n)$, T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ (see, for example, [**11**]). Also, the following results are known.

Theorem A. *Suppose that $A_t = tE$ and $r(x) = |x|$, where E denotes the identity matrix and $|x|$ denotes the Euclidean norm for x; also suppose that $K_0 \in L \log L(\mathbb{R}^n)$. The operator T is then of weak-type* $(1, 1)$.

Theorem B. *Suppose that*

$$A_t x = (t^{\alpha_1} x_1, t^{\alpha_2} x_2, \dots, t^{\alpha_n} x_n),$$

where $x = (x_1, \dots, x_n)$ and $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n$. Also, suppose that $\Sigma = S^{n-1} = \{|x| = 1\}$ and $K_0 \in L \log L(\mathbb{R}^n)$. Then T is of weak-type $(1, 1)$.

Theorem A is due to Seeger [**12**]. In low-dimensional cases, a version of Theorem A was proved in [**4**, **6**]. (See [**3**, **5**, **7**, **10**, **13**, **15**, **16**] for relevant results.) Theorem B is a particular case of a result of Tao [**15**]. In [**15**], the weak-type $(1, 1)$ boundedness was proved for singular integrals on general homogeneous groups. Note that the proof given in [**15**] does not use the Fourier transform.

Remark 1.1. In Theorem B, the assumption that $\Sigma = S^{n-1}$ can be relaxed. We note that the method of [**15**] can prove a version of Theorem B where Σ is only assumed to be an ellipsoid in statement (iii) above. We use this fact in § 8.

In this paper we prove the following result.

Theorem 1.2. *Suppose that $n = 2$ and $K_0 \in L \log L(\mathbb{R}^n)$. The operator T is then of weak-type* $(1, 1)$.

There exists a non-singular real matrix Q such that $Q^{-1}PQ$ is one of the following Jordan canonical forms:

$$P_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}, \qquad P_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \tag{1.2}$$

where $\alpha, \beta > 0$. Since the case where $P = P_1$ is handled by Theorem B and Remark 1.1, to prove Theorem 1.2 we must consider the cases $P = P_2$ and $P = P_3$. In § 8, we shall give an argument that derives Theorem 1.2 from results for P having the form of (1.2).

In § 2, we give an outline of a proof of Theorem 1.2. We shall see that Theorem 1.2 follows from Proposition 2.2. A proof of Proposition 2.2 for P_2 will be given in §§ 3–6. We shall give a proof of Proposition 2.2 for P_3 in § 7. The framework of our proof of Theorem 1.2 is similar to that of Theorem B in [**15**], but we need some new arguments in §§ 4–8, which do not occur in [**15**]. In Appendix A, for completeness we shall give proofs of four results of §§ 2 and 3 by applying the methods of [**15**]. Although we assume $n = 2$ in §§ 3–8, several results can extend to higher dimensions. In this paper, C, C_1, C_2 will be used to denote non-negative constants which may be different in different occurrences.

2. Outline of proof of Theorem 1.2

We normalize $\|K_0\|_{L \log L} = 1$, where K_0 is as in (1.1). We may assume that K is real valued. Let $\delta_t f(x) = t^{-\gamma} f(A_t^{-1} x)$. Then

$$K(x) = \frac{1}{\log 2} \int_0^\infty \frac{\delta_t K_0(x)\,\mathrm{d}t}{t}.$$

Let φ be a non-negative function in $C_0^\infty(\mathbb{R})$ supported in $[\frac{1}{2}, 2]$ such that

$$\sum_{j=-\infty}^{\infty} 2^{-j} t \varphi(2^{-j} t) = \frac{1}{\log 2} \quad \text{for } t \neq 0.$$

We decompose K as $K = \sum_{j=-\infty}^{\infty} S_j K_0$, where

$$S_j f = 2^{-j} \int_0^\infty \varphi(2^{-j} t) \delta_t f \,\mathrm{d}t.$$

We note that

$$\|S_j f\|_1 \leqslant C \|f\|_1, \tag{2.1}$$

where C is independent of j.

Let B be a subset of $\mathbb{R}^n$ such that

$$B = \{x \in \mathbb{R}^n \colon r(x - a) < s\}$$

for some $a \in \mathbb{R}^n$ and $s > 0$. Then we call B a ball with centre a and radius s and we write $B = B(a, s)$. If $s = 2^k$ for some $k \in \mathbb{Z}$ (the set of all integers), then $B(a, 2^k)$ is called a dyadic ball. Also, we write $a = x_B$, $k = k(B)$. Let $CB(a, s) = B(a, Cs)$ for $C > 0$.

We have to show that

$$|\{x \in \mathbb{R}^n \colon |Tf(x)| > \lambda\}| \leqslant C \lambda^{-1} \|f\|_1 \quad \text{for all } \lambda > 0,$$

when $\|K_0\|_{L \log L} = 1$. We may assume that $\lambda = 1$. By Calderón–Zygmund decomposition of f at height 1, we have

$$f = g + \sum_B b_B,$$

where the balls B range over a collection of disjoint dyadic balls and

$$\|g\|_1 \leqslant C\|f\|_1, \qquad \|g\|_\infty \leqslant C, \tag{2.2}$$

$$\sum_B |B| \leqslant C\|f\|_1, \tag{2.3}$$

$$\operatorname{supp}(b_B) \subset CB, \tag{2.4}$$

$$\|b_B\|_1 \leqslant C|B|, \tag{2.5}$$

$$\int b_B = 0. \tag{2.6}$$

We may assume that the functions b_B are real valued and smooth. Also, we may assume that the family of the balls $\{B\}$ is finite. We have

$$\{x \in \mathbb{R}^n \colon |Tf(x)| > 1\} \subset G_1 \cup G_2 \cup G_3,$$

where

$$\begin{aligned}
G_1 &= \{x \in \mathbb{R}^n \colon |Tg(x)| > \tfrac{1}{3}\},\\
G_2 &= \Big\{x \in \mathbb{R}^n \colon \sum_{s \leqslant C} \Big| \sum_B (b_B * S_{k(B)+s}K_0)(x)\Big| > \tfrac{1}{3}\Big\},\\
G_3 &= \Big\{x \in \mathbb{R}^n \colon \sum_{s > C} \Big| \sum_B (b_B * S_{k(B)+s}K_0)(x)\Big| > \tfrac{1}{3}\Big\}.
\end{aligned}$$

Here C is a sufficient large positive constant. Since T is bounded on L^2, by Chebyshev's inequality and (2.2) we have

$$|G_1| \leqslant C\|g\|_2^2 \leqslant C\|g\|_1 \leqslant C\|f\|_1.$$

The set G_2 is contained in $E = \bigcup_B C_1 B$ for some $C_1 > 0$, since we have (2.4) and $\operatorname{supp}(S_j K_0)$ is contained in $\{2^{j-1} \leqslant r(x) \leqslant 2^{j+2}\}$. So,

$$|G_2| \leqslant |E| \leqslant C\|f\|_1$$

by (2.3). Therefore, to prove Theorem 1.2 it remains to show that $|G_3| \leqslant C\|f\|_1$. This follows from the estimate

$$\Big|\Big\{x \in \mathbb{R}^n \colon \sum_{s > C} \Big| \sum_B \psi_{2^s B}(x)(b_B * S_{k(B)+s}K_0)(x)\Big| > \tfrac{1}{3}\Big\}\Big| \leqslant C_1 \sum_B |B|, \tag{2.7}$$

where the function ψ_B is defined as

$$\psi_B(x) = \psi_0(A_{2^{-k(B)}}(x - x_B))$$

with a non-negative function ψ_0 in $C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\psi_0) \subset \{d_1^{-1} \leqslant r(x) \leqslant d_1\}$, $\psi_0(x) = 1$ if $2/d_1 \leqslant r(x) \leqslant d_1/2$ for a sufficiently large positive number d_1 and $\|\psi_0\|_\infty \leqslant 1$.

Let $\mathcal{B}$ be a finite family of disjoint dyadic balls B such that

$$\sum_{B\in\mathcal{B}} |B| \leqslant 1. \tag{2.8}$$

As in [**15**], the following result implies (2.7) (see § A.1).

Proposition 2.1. *Let $1 < p < 2$ and $s > C$, where C is as in (2.7). Let $\mathcal{B}$ be as in (2.8). For each $B \in \mathcal{B}$, let b_B be a smooth real-valued function satisfying (2.4)–(2.6). There then exist a positive number ϵ and an exceptional set E_s such that*

$$|E_s| \leqslant C2^{-\epsilon s} \tag{2.9}$$

and

$$\Big\| \sum_{B\in\mathcal{B}} \psi_{2^s B}(b_B * S_{k(B)+s} f_B) \Big\|_{L^p(E_s^{\mathrm{c}})} \leqslant C2^{-\epsilon s} \Big(\sum_{B\in\mathcal{B}} |B| \, \|f_B\|_2^2 \Big)^{1/2} \tag{2.10}$$

for all real-valued functions f_B in $L^2(\mathbb{R}^n)$, where E_s^{c} denotes the complement of E_s.

Also, as in [**15**], Proposition 2.1 can be derived from the following.

Proposition 2.2. *Let p, s, $\mathcal{B}$ and $\{b_B\}_{B\in\mathcal{B}}$ be as in Proposition 2.1. There then exist constants $C_1 > 1$ and $\epsilon > 0$ such that if*

$$\Big\| \sum_{B\in\mathcal{B}} \chi_{C_1 2^s B} \Big\|_\infty \leqslant C2^{\gamma s}, \tag{2.11}$$

then we have

$$\Big\| \sum_{B\in\mathcal{B}} \psi_{2^s B}(b_B * S_{k(B)+s} f_B) \Big\|_p \leqslant C2^{-\epsilon s} \Big(\sum_{B\in\mathcal{B}} |B| \, \|f_B\|_2^2 \Big)^{1/2} \tag{2.12}$$

for all real-valued functions f_B in $L^2(\mathbb{R}^n)$.

To prove Propositions 2.1 and 2.2, we use the following version of [**15**, Lemma 9.2].

Lemma 2.3. *Let C_1, C_2, C_3 be positive constants. Let $S = B(x_S, u_S)$, $u_S = C_1 2^{-\delta s}$, $0 \leqslant \delta \leqslant 1$, and $r(x_S) < C_2$, where s is a positive integer. Define*

$$\psi_{B,S}(x) = \varPsi_S(A_{2^{-k(B)-s}}(x - x_B)), \tag{2.13}$$

where $\varPsi_S(x) = \varPsi(A_{u_S^{-1}}(x - x_S))$ with a fixed non-negative function $\varPsi$ in C_0^∞ such that $\|\varPsi\|_\infty \leqslant 1$, $\operatorname{supp}(\varPsi) \subset \{r(x) \leqslant 1\}$ and $\varPsi(x) = 1$ if $r(x) \leqslant \frac{1}{2}$. Then we have

$$\Big| \Big\{ x \in \mathbb{R}^n \colon \sum_{B\in\mathcal{B}} \psi_{B,S}(x) > C_3 s^3 2^{\gamma s} |S| \Big\} \Big| \leqslant C2^{-cs^2},$$

where c is a positive constant and $\mathcal{B}$ is as in Proposition 2.1.

See § A.2 for a proof of Lemma 2.3 and § A.3 for a proof of Proposition 2.1 using Proposition 2.2 and Lemma 2.3.

Remark 2.4. From Proposition 2.1 and arguments in [**5**], we can prove some weighted weak-type $(1, 1)$ estimates for the singular integral operator T under certain conditions.

3. Proof of Proposition 2.2: preliminaries

To prove Theorem 1.2, it remains to show Proposition 2.2. To obtain (2.12), by duality it suffices to show that

$$\bigg(\sum_{B\in\mathcal{B}} |B|^{-1}\|S^*_{k(B)+s}(\tilde{b}_B * (\psi_{2^sB}F))\|_2^2\bigg)^{1/2} \leqslant C2^{-\epsilon s}\|F\|_{p'} \tag{3.1}$$

for real-valued functions F, where $p' = p/(p-1)$, $\tilde{b}_B(x) = b_B(-x)$ and S_j^* is the adjoint of S_j:

$$S_j^*G(x) = 2^{-j}\int_0^\infty \varphi(2^{-j}t)G(A_tx)\,\mathrm{d}t.$$

To prove (3.1), by the TT^* method, it suffices to show that

$$\bigg\|\sum_{B\in\mathcal{B}} |B|^{-1}\psi_{2^sB}(b_B * S_{k(B)+s}S^*_{k(B)+s}(\tilde{b}_B * (\psi_{2^sB}F)))\bigg\|_p \leqslant C2^{-2\epsilon s}\|F\|_{p'}. \tag{3.2}$$

Note that

$$S_{j+s}S^*_{j+s} = 2^{-\gamma(j+s)}S_0S_0^*.$$

Therefore, we can rewrite (3.2) as

$$\|TF\|_p \leqslant C2^{-2\epsilon s}\|F\|_{p'}, \qquad T = 2^{-\gamma s}\sum_{B\in\mathcal{B}} \psi_{2^sB}T_B\psi_{2^sB}, \tag{3.3}$$

where T_B is the self-adjoint operator defined as

$$T_BF = |B|^{-1}b_B * S_0S_0^*(|B|^{-1}\tilde{b}_B * F).$$

Define the smooth function a_B supported on the ball $B(0, C)$ by

$$a_B(v) = b_B(d_B(v)),$$

where d_B is the mapping defined as

$$d_B(v) = x_B + A_{2^{k(B)}}v. \tag{3.4}$$

Then by (2.4)–(2.6) we see that

$$\operatorname{supp}(a_B) \subset B(0, C), \qquad \|a_B\|_1 \leqslant C, \qquad \int a_B(v)\,\mathrm{d}v = 0. \tag{3.5}$$

Also, note that

$$S_0S_0^*F(x) = \int_0^\infty \tilde{\varphi}(t)F(A_tx)\,\mathrm{d}t,$$

where $\tilde{\varphi}$ is a non-negative function in C_0^∞ with support in $[\frac{1}{4}, 4]$. Thus, we can rewrite the operator T_B, up to a constant factor, as

$$T_BF(x) = \iiint a_B(v)\tilde{\varphi}(t)a_B(w)F(d_B(w) + A_t(x - d_B(v)))\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}t. \tag{3.6}$$

We need the following result [**15**].

Lemma 3.1. *Let f be a continuous function on $\mathbb{R}^2$ such that*

$$\operatorname{supp}(f) \subset B(0, C_1), \qquad \int f(x)\,\mathrm{d}x = 0, \qquad \|f\|_1 \leqslant C_2.$$

Then there exist functions f_1, f_2 such that

$$f(x) = \sum_{i=1}^{2} \partial_{x_i} f_i(x),$$
$$\operatorname{supp}(f_i) \subset B(0, C_1'), \quad \|f_i\|_1 \leqslant C_2' \quad \text{for } i = 1, 2,$$

for some constants C_1' and C_2' with $C_1' \geqslant C_1$.

Let

$$\psi_B^+(x) = \psi^+(A_{2^{-k(B)}}(x - x_B)),$$

where ψ^+ is a non-negative function in $C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\psi^+) \subset \{d_2^{-1} \leqslant r(x) \leqslant d_2\}$ and $\psi^+(x) = 1$ if $2/d_2 \leqslant r(x) \leqslant d_2/2$, where d_2 is a constant satisfying $d_2 > 2d_1$. We note that ψ_B^+ is positive on the support of ψ_B. Let $C_1 \geqslant d_2$, where C_1 is as in (2.11). By Lemma 3.1 we can find functions a_B^1, a_B^2 supported on $B(0, C)$ such that

$$a_B = \sum_{i=1}^{2} \partial_{x_i} a_B^i(x), \quad \|a_B^i\|_1 \leqslant C \quad \text{for } i = 1, 2. \tag{3.7}$$

Let

$$a_B^+ = |a_B| + \sum_{i=1}^{2} |a_B^i|.$$

Then

$$a_B^+ \geqslant 0, \qquad \operatorname{supp}(a_B^+) \subset B(0, C), \qquad \|a_B^+\|_1 \leqslant C. \tag{3.8}$$

Let φ^+ be a non-negative function in C_0^∞ such that $\operatorname{supp}(\varphi^+) \subset [\frac{1}{8}, 8]$, $\varphi^+ > 0$ on $\operatorname{supp}(\tilde{\varphi})$ and $\varphi^+(t) = t^{\gamma-2}\varphi^+(t^{-1})$. Define the self-adjoint operator T_B^+ by

$$T_B^+ F(x) = \iiint a_B^+(v)\varphi^+(t) a_B^+(w) F(d_B(w) + A_t(x - d_B(v)))\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}t.$$

Set

$$T^+ = 2^{-\gamma s} \sum_{B \in \mathcal{B}} \psi_{2^s B}^+ T_B^+ \psi_{2^s B}^+. \tag{3.9}$$

Then

$$|T_B F(x)| \leqslant C T_B^+ F(x) \quad \text{for all } B, \qquad |TF(x)| \leqslant C T^+ F(x),$$

if F is non-negative.

As in [**15**], we can show that

$$\|T^+ F\|_p \leqslant C \|F\|_q \quad \text{for all } 1 \leqslant p \leqslant q \leqslant \infty \tag{3.10}$$

under the condition $C_1 \geqslant d_2$, where C_1 is as in (2.11) and d_2 is as in the definition of ψ_B^+ (see § A.4).

The estimate (3.3) follows from

$$\|T^2F\|_p \leqslant C2^{-\epsilon s}\|F\|_{p'} \quad \text{for some } \epsilon > 0. \tag{3.11}$$

To see this, by the TT^* method, the self-adjointness of T and (3.11) we first note that

$$\|TF\|_p \leqslant C2^{-\epsilon s/2}\|F\|_2. \tag{3.12}$$

Next, by (3.10) we have $\|TF\|_p \leqslant C\|F\|_q$, $1 \leqslant p \leqslant q \leqslant \infty$. Interpolating between this and (3.12) under the condition $1 < p < 2$, we have (3.3) for some $\epsilon > 0$.

It remains to prove (3.11). Since $T^2 \colon L^2 \to L^2$ by (3.10), it suffices to prove (3.11) for $p = 1$ if we take into account interpolation. Expanding T^2, we thus have to prove

$$\left\| 2^{-2\gamma s} \sum_{B_1, B_2 \in \mathcal{B}} \left(\prod_{i=1}^{2} \psi_{2^s B_i} T_{B_i} \psi_{2^s B_i} \right) F \right\|_1 \leqslant C2^{-\epsilon s}\|F\|_\infty.$$

By duality and self-adjointness this follows from

$$2^{-2\gamma s} \sum_{B \in \mathcal{B}_0} \left| \left\langle \left(\prod_{i=1}^{2} \psi_{2^s B_i} T_{B_i} \psi_{2^s B_i} \right) F_B, G_B \right\rangle \right| \leqslant C2^{-\epsilon s} \tag{3.13}$$

for all real-valued smooth functions F_B, G_B satisfying $\|F_B\|_\infty \leqslant 1$, $\|G_B\|_\infty \leqslant 1$, where

$$\mathcal{B}_0 = \{B = (B_1, B_2) \in \mathcal{B}^2 \colon k(B_1) \leqslant k(B_2)\}. \tag{3.14}$$

The inner product in (3.13) can be written, up to a constant factor, as

$$\iiiint G_B(x_0)F_B(x_2)H_B(x_0, x_1, x_2, t, v, w)\,\mathrm{d}x_0\,\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}v; \tag{3.15}$$

thus,

$$H_B(x_0, x_1, x_2, t, v, w) = \prod_{i=1}^{2} (\psi_{2^s B_i}(x_{i-1}) a_{B_i}(v_i) \tilde{\varphi}(t_i) a_{B_i}(w_i) \psi_{2^s B_i}(x_i)),$$

where $x_0 \in \mathbb{R}^2$, $v = (v_1, v_2) \in \mathbb{R}^2 \times \mathbb{R}^2$, $w = (w_1, w_2) \in \mathbb{R}^2 \times \mathbb{R}^2$, $t = (t_1, t_2) \in (0, \infty) \times (0, \infty)$ and we may assume that $v, w \in B(0, C)^2$, $t \in [C^{-1}, C]^2$; $\mathrm{d}w = \mathrm{d}w_1\,\mathrm{d}w_2$, $\mathrm{d}v = \mathrm{d}v_1\,\mathrm{d}v_2$, $\mathrm{d}t = \mathrm{d}t_1\,\mathrm{d}t_2$; x_1, x_2 are defined as follows:

$$x_1 = d_{B_1}(w_1) + A_{t_1}(x_0 - d_{B_1}(v_1)), \qquad x_2 = d_{B_2}(w_2) + A_{t_2}(x_1 - d_{B_2}(v_2)). \tag{3.16}$$

We note that each x_i, $i = 1, 2$, is a function of x_0 and B_ℓ, v_ℓ, w_ℓ, t_ℓ for all ℓ with $1 \leqslant \ell \leqslant i$. We also write $y = (y_1, y_2) = v_1 \in \mathbb{R}^2$.

4. Proof of Proposition 2.2 for P_2: basic estimates

Suppose that $P = P_2$, where P_2 is as in (1.2). Then

$$A_t = t^{\alpha} \begin{pmatrix} 1 & 0 \\ \log t & 1 \end{pmatrix}.$$

Let

$$M_B = 2^{\alpha(k(B_1)+s)} 2^{\alpha(k(B_2)+s)} (1 + |k(B_1) - k(B_2)|) \tag{4.1}$$

for $B = (B_1, B_2) \in \mathcal{B}^2$. Let $D_t(x_2)$ be the matrix such that the first column vector is $\partial_{t_1} x_2$ and the second column vector is $\partial_{t_2} x_2$, where x_2 is as in (3.16). The following two estimates imply (3.13):

$$\sum_{B \in \mathcal{B}_0} \left| \iiiint G_B(x_0) F_B(x_2) H_B \zeta_1(2^{\delta s} M_B^{-1} \det(D_t(x_2)))\, \mathrm{d}x_0\, \mathrm{d}w\, \mathrm{d}t\, \mathrm{d}v \right| \leqslant C 2^{-\epsilon s} 2^{2\gamma s}, \tag{4.2}$$

$$\sum_{B \in \mathcal{B}_0} \left| \iiiint G_B(x_0) F_B(x_2) H_B \zeta_2(2^{\delta s} M_B^{-1} \det(D_t(x_2)))\, \mathrm{d}x_0\, \mathrm{d}w\, \mathrm{d}t\, \mathrm{d}v \right| \leqslant C 2^{-\epsilon s} 2^{2\gamma s}, \tag{4.3}$$

where H_B is as in (3.15); ζ_1 is a non-negative function in $C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}(\zeta_1) \subset [-1, 1]$, $\zeta_1(t) = 1$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$; $\zeta_2 = 1 - \zeta_1$; δ is a small positive number to be specified in the following.

Let $D_{y_i,t_j}(x_2)$ be the matrix such that the first column vector is $\partial_{y_i} x_2$ and the second column vector is $\partial_{t_j} x_2$ for $i, j = 1, 2$. To prove (4.2) and (4.3) we use the following lemma and results in its proof.

Lemma 4.1. *Let M_B be as in (4.1). Suppose that $B \in \mathcal{B}_0$, where $\mathcal{B}_0$ is as in (3.14), and that $t_\ell \in [C^{-1}, C]$, $x_{\ell-1} \in \mathrm{supp}(\psi^+_{2^s B_\ell})$, $v_\ell \in B(0, C)$, $\ell = 1, 2$, where x_1 is as in (3.16). Then we have the following:*

$$|\det(D_t(x_2))| + s^{-1} 2^{\alpha s} |\partial_{y_i} \det(D_t(x_2))| + |\partial_{t_j} \det(D_t(x_2))| \leqslant C M_B, \tag{4.4}$$

$$s^{-1} 2^{\alpha s} |\det(D_{y_i,t_j}(x_2))| + s^{-1} 2^{\alpha s} |\partial_{t_k} \det(D_{y_i,t_j}(x_2))| \leqslant C M_B \tag{4.5}$$

for $i, j, k = 1, 2$, and

$$|\psi_{2^s B_\ell}(x_{\ell'})| + s^{-1} 2^{\alpha s} |\partial_{y_i} \psi_{2^s B_\ell}(x_{\ell'})| + |\partial_{t_j} \psi_{2^s B_\ell}(x_{\ell'})| \leqslant C \psi^+_{2^s B_\ell}(x_{\ell'}) \tag{4.6}$$

for $i, j = 1, 2$, $0 \leqslant \ell' \leqslant \ell$, $\ell = 1, 2$.

Proof. We note the following formulae, which hold for general $A_t = t^P$:

$$\begin{aligned}
\partial_{t_\ell} x_k &= t_\ell^{-1} P A_{t_\ell \cdots t_k}(x_{\ell-1} - d_{B_\ell}(v_\ell)) \quad \text{if } \ \ell \leqslant k, && (4.7)\\
\partial_{t_\ell} x_k &= 0 \quad \text{if } \ell > k, && (4.8)\\
\partial_{t_1}^2 x_2 &= -t_1^{-2} P A_{t_1 t_2}(x_0 - d_{B_1}(v_1)) + t_1^{-2} P^2 A_{t_1 t_2}(x_0 - d_{B_1}(v_1)), && (4.9)\\
\partial_{t_1}\partial_{t_2} x_2 &= \partial_{t_2}\partial_{t_1} x_2 = t_1^{-1} t_2^{-1} P^2 A_{t_1 t_2}(x_0 - d_{B_1}(v_1)), && (4.10)\\
\partial_{t_2}^2 x_2 &= -t_2^{-2} P A_{t_2}(x_1 - d_{B_2}(v_2)) + t_2^{-2} P^2 A_{t_2}(x_1 - d_{B_2}(v_2)), && (4.11)\\
\partial_{y_i} x_\ell &= -A_{t_1 \cdots t_\ell 2^{k(B_1)}} e_i, \quad i, \ell = 1, 2, && (4.12)\\
\partial_{t_1}\partial_{y_i} x_1 &= -t_1^{-1} P A_{t_1 2^{k(B_1)}} e_i, \quad \partial_{t_2}\partial_{y_i} x_1 = 0, \quad i = 1, 2, && (4.13)\\
\partial_{t_j}\partial_{y_i} x_2 &= -t_j^{-1} P A_{t_1 t_2 2^{k(B_1)}} e_i, \quad i, j = 1, 2, && (4.14)
\end{aligned}$$

where $\{e_i\}$ is the standard basis. Let

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then

$$\det(D_t(x_2)) = \langle \partial_{t_1} x_2, L \partial_{t_2} x_2 \rangle = \langle X, A^*_{2^{k(B_1)+s}} L A_{2^{k(B_2)+s}} Y \rangle, \tag{4.15}$$

where $X = A_{2^{-k(B_1)-s}} \partial_{t_1} x_2$, $Y = A_{2^{-k(B_2)-s}} \partial_{t_2} x_2$. We note that

$$A^*_{2^h} L A_{2^m} = 2^{h\alpha} 2^{m\alpha} \begin{pmatrix} (m-h)\log 2 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.16}$$

By the assumptions and (4.7), we have $|X| \leqslant C$ and $|Y| \leqslant C$. Thus, by (4.15) and (4.16), we have

$$|\det(D_t(x_2))| \leqslant C M_B.$$

Similarly by (4.7), (4.15), (4.16), (4.9)–(4.11) we have

$$|\partial_{t_j} \det(D_t(x_2))| \leqslant C M_B,$$

since $k(B_1) \leqslant k(B_2)$.

Next, by (4.14) we have

$$\langle \partial_{t_1} x_2, L \partial_{t_2} \partial_{y_i} x_2 \rangle = -t_2^{-1} \langle X, A^*_{2^{k(B_1)+s}} L A_{2^{k(B_1)}} A_{t_1 t_2} P e_i \rangle,$$

where X is as above. Thus, by (4.7) and (4.16) we have

$$|\langle \partial_{t_1} x_2, L \partial_{t_2} \partial_{y_i} x_2 \rangle| \leqslant C s 2^{\alpha k(B_1)} 2^{\alpha(k(B_1)+s)} \leqslant C s 2^{-\alpha s} M_B,$$

since $k(B_1) \leqslant k(B_2)$. Also, by (4.14) we have

$$\langle \partial_{t_1} \partial_{y_i} x_2, L \partial_{t_2} x_2 \rangle = -t_1^{-1} \langle P A_{t_1 t_2} e_i, A^*_{2^{k(B_1)}} L A_{2^{k(B_2)+s}} Y \rangle,$$

where Y is as above. Therefore, arguing as above, we have

$$\begin{aligned}|\langle \partial_{t_1}\partial_{y_i}x_2, L\partial_{t_2}x_2\rangle| &\leqslant C(s+|k(B_2)-k(B_1)|)2^{\alpha k(B_1)}2^{\alpha(k(B_2)+s)}\\ &\leqslant Cs2^{-\alpha s}M_B.\end{aligned}$$

From these estimate it follows that

$$|\partial_{y_i}\det(D_t(x_2))| \leqslant Cs2^{-\alpha s}M_B.$$

Collecting results, we obtain (4.4).

Similarly, by (4.12) and (4.7) we see that

$$\begin{aligned}|\det(D_{y_i,t_j}(x_2))| &\leqslant C(s+|k(B_1)-k(B_j)|)2^{\alpha k(B_1)}2^{\alpha(k(B_j)+s)}\\ &\leqslant Cs2^{-\alpha s}M_B.\end{aligned} \tag{4.17}$$

By (4.14) and (4.7) we have

$$\begin{aligned}|\langle \partial_{t_k}\partial_{y_i}x_2, L\partial_{t_j}x_2\rangle| &\leqslant C(s+|k(B_j)-k(B_1)|)2^{\alpha(k(B_j)+s)}2^{\alpha k(B_1)}\\ &\leqslant Cs2^{-\alpha s}M_B.\end{aligned} \tag{4.18}$$

If $m=\min(k,j)$, from (4.9)–(4.12) it follows that

$$\begin{aligned}|\langle \partial_{y_i}x_2, L\partial_{t_k}\partial_{t_j}x_2\rangle| &\leqslant C(s+|k(B_m)-k(B_1)|)2^{\alpha(k(B_m)+s)}2^{\alpha k(B_1)}\\ &\leqslant Cs2^{-\alpha s}M_B.\end{aligned} \tag{4.19}$$

The estimates (4.18) and (4.19) imply

$$|\partial_{t_k}\det(D_{y_i,t_j}(x_2))| \leqslant Cs2^{-\alpha s}M_B. \tag{4.20}$$

Thus, (4.5) follows from (4.17) and (4.20).

To prove (4.6), we recall that $\psi_{2^sB_\ell}(x_{\ell'})=\psi_0(A_{2^{-k(B_\ell)-s}}(x_{\ell'}-x_{B_\ell}))$. By (4.12) we have

$$\partial_{y_i}A_{2^{-k(B_\ell)-s}}(x_{\ell'}-x_{B_\ell}) = -A_{2^{-k(B_\ell)-s}}A_{t_1\cdots t_{\ell'}2^{k(B_1)}}e_i, \quad \ell'=1,2.$$

Therefore,

$$|\partial_{y_i}A_{2^{-k(B_\ell)-s}}(x_{\ell'}-x_{B_\ell})| \leqslant Cs2^{-\alpha s}. \tag{4.21}$$

By (4.7) and (4.8) we see that

$$\partial_{t_j}A_{2^{-k(B_\ell)-s}}(x_{\ell'}-x_{B_\ell}) = \begin{cases} t_j^{-1}A_{2^{-k(B_\ell)-s}}PA_{t_j\cdots t_{\ell'}}(x_{j-1}-d_{B_j}(v_j)) & \text{if } 1\leqslant j\leqslant \ell',\\ 0 & \text{if } j>\ell'.\end{cases}$$

Also, we note that

$$|A_{2^{-k(B_j)-s}}(x_{j-1}-d_{B_j}(v_j))| \leqslant C, \quad j=1,2,$$

by the assumptions. Therefore, we have

$$|\partial_{t_j}A_{2^{-k(B_\ell)-s}}(x_{\ell'}-x_{B_\ell})| \leqslant C, \tag{4.22}$$

since $k(B_j)\leqslant k(B_\ell)$ if $1\leqslant j\leqslant \ell'\leqslant \ell$. From (4.21), (4.22) and the chain rule, we have (4.6). $\square$

5. Proof of Proposition 2.2 for P_2: proof of (4.2)

In this section we prove (4.2). It suffices to show that

$$\sum_{B\in\mathcal{B}_0}\iiint\prod_{i=1}^{2}(\psi^+_{2^sB_i}(x_{i-1})a^+_{B_i}(v_i)a^+_{B_i}(w_i))\zeta_1(2^{\delta s}M_B^{-1}\det(D_t(x_2)))\,\mathrm{d}x_0\,\mathrm{d}w\,\mathrm{d}v \leqslant C2^{-\epsilon s}2^{2\gamma s} \quad (5.1)$$

uniformly in $t_i\in[C^{-1},C]$ for $i=1,2$. We fix t.

Let
$$\tilde{\psi}^+_B(x)=\tilde{\psi}^+(A_{2^{-k(B)}}(x-x_B)),$$
where $\tilde{\psi}^+$ is a non-negative function in $C_0^\infty(\mathbb{R}^n)$ such that
$$\mathrm{supp}(\tilde{\psi}^+)\subset\{d_3^{-1}\leqslant r(x)\leqslant d_3\},$$
$\tilde{\psi}^+(x)=1$ if $2/d_3\leqslant r(x)\leqslant d_3/2$. We assume that $d_3>2d_2$, where d_2 is as in the definition of ψ^+_B. Let $S=B(x_S,2^{-\delta_0 s})\subset B(0,C)$, $0<\delta_0<1$, where the positive integer s is as in (5.1). Let $\psi_{B,S}$ be as in Lemma 2.3. Define

$$U_S(x)=\sum_{B\in\mathcal{B},\,x\in\mathrm{supp}\,\tilde{\psi}^+_{2^sB}}\psi_{B,S}(x). \quad (5.2)$$

For $x\in\mathbb{R}^2$ we consider the condition

$$U_S(x)\leqslant s^3 2^{\gamma s}|S| \quad \text{for all balls } S=B(x_S,2^{-\delta_0 s})\subset B(0,C), \quad (5.3)$$

where the positive number δ_0 and the ball $B(0,C)$ will be specified below. Then we have the following version of [**15**, Lemma 12.2].

Lemma 5.1. *Let $E=\{x\in\mathbb{R}^2\colon x$ does not satisfy (5.3)$\}$. Then*

$$|E|\leqslant C2^{-\epsilon_0 s^2}$$

for some $\epsilon_0>0$.

To prove Lemma 5.1 we use the following covering lemma [**1**].

Lemma 5.2. *Let $\mathcal{G}=\{B(a_\lambda,u_\lambda)\colon\lambda\in\Lambda\}$ be a family of balls such that $\sup_{\lambda\in\Lambda}u_\lambda<\infty$. There is then a subfamily $\mathcal{G}'=\{B(c_j,r_j)\colon j=1,2,\dots\}$ of $\mathcal{G}$ such that $\mathcal{G}'$ is at most countable, balls in $\mathcal{G}'$ are disjoint and for any $B(a_\lambda,u_\lambda)\in\mathcal{G}$ we can find a ball $B(c_j,r_j)\in\mathcal{G}'$ satisfying $B(a_\lambda,u_\lambda)\subset B(c_j,dr_j)$ for some positive constant d independent of $\mathcal{G}$.*

Proof of Lemma 5.1. By applying Lemma 5.2 to the family of balls

$$\mathcal{G}=\{S=B(x_S,2^{-\delta_0 s})\colon S\subset B(0,C)\},$$

we have a subfamily of disjoint balls $\{S_i\}_{i=1}^N$ in $B(0,C)$, $N \leqslant C2^{s\delta_0\gamma}$, such that if $\tilde{S}_i = C_1 S_i$ with a constant $C_1 \geqslant 2d$, for any S in $\mathcal{G}$ there exists $i \in \{1,2,\dots,N\}$ for which it holds that

$$\psi_{B,S}(x) \leqslant \psi_{B,\tilde{S}_i}(x) \quad \text{for all } B, \tag{5.4}$$

where $\psi_{B,\tilde{S}_i}$ is defined as in (2.13) with $\tilde{S}_i$ in place of S. From (5.4) it follows that

$$U_S(x) \leqslant U_{\tilde{S}_i}(x) \quad \text{for some } i \in \{1,2,\dots,N\}, \tag{5.5}$$

where $U_{\tilde{S}_i}$ is defined as in (5.2) with $\tilde{S}_i$ in place of S. We see that (5.5) implies

$$E \subset \bigcup_{i=1}^{N} \{x \colon U_{\tilde{S}_i}(x) \geqslant Cs^3 2^{\gamma s} |\tilde{S}_i|\}.$$

Therefore, the conclusion follows from an application of Lemma 2.3. $\square$

Let the set E be as in Lemma 5.1. Writing

$$1 = (\chi_E(x_0) + \chi_{E^c}(x_0))(\chi_E(x_1) + \chi_{E^c}(x_1))$$

and expanding the right-hand side, by (3.8) we can see that to prove (5.1) it suffices to show the following two estimates:

$$\sum_B \iiint \prod_{i=1}^{2} \psi^+_{2^s B_i}(x_{i-1}) \chi_E(x_\ell) a^+_{B_1}(v_1) a^+_{B_1}(w_1)\, \mathrm{d}x_0\, \mathrm{d}v_1\, \mathrm{d}w_1 \leqslant C 2^{-\epsilon s} 2^{2\gamma s} \tag{5.6}$$

for $\ell = 0,1$, where we note that x_1 is independent of v_2 and w_2, and

$$\sum_B \iiint_{|\det(D_t(x_2))| \leqslant 2^{-\delta s} M_B} \prod_{i=1}^{2} (\psi^+_{2^s B_i}(x_{i-1}) \chi_{E^c}(x_{i-1}) a^+_{B_i}(v_i) a^+_{B_i}(w_i))\, \mathrm{d}x_0\, \mathrm{d}v\, \mathrm{d}w \leqslant C 2^{-\epsilon s} 2^{2\gamma s} \tag{5.7}$$

for some $\epsilon > 0$, where the balls B range over $\mathcal{B}_0$.

Proof of (5.6). First, let $\ell = 0$. Since $C_1 \geqslant d_2$, where C_1 is as in Proposition 2.2 and d_2 is as in the definition of ψ^+_B, by (2.11) and (3.8), the left-hand side of (5.6) is bounded by I, where

$$\mathrm{I} = C 2^{\gamma s} \sum_{B_1} \int \psi^+_{2^s B_1}(x_0) \chi_E(x_0)\, \mathrm{d}x_0.$$

By (2.11) and Lemma 5.1, we have

$$\mathrm{I} \leqslant C 2^{2\gamma s} \int \chi_E(x_0)\, \mathrm{d}x_0 \leqslant C 2^{2\gamma s} |E| \leqslant C 2^{2\gamma s} 2^{-\epsilon_0 s^2}.$$

Next, let $\ell = 1$. As above, by (2.11) the left-hand side of (5.6) is bounded by II, where

$$\mathrm{II} = C 2^{\gamma s} \sum_{B_1} \iiint \psi^+_{2^s B_1}(x_0) \chi_E(x_1) a^+_{B_1}(v_1) a^+_{B_1}(w_1)\, \mathrm{d}x_0\, \mathrm{d}v_1\, \mathrm{d}w_1.$$

By a change of variables, we see that

$$\int \psi^+_{2^s B_1}(x_0)\chi_E(x_1)\,\mathrm{d}x_0 = t_1^{-\gamma}\int \psi^+_{2^s B_1}(\tilde{x}_0)\chi_E(x_0)\,\mathrm{d}x_0,$$

where

$$\tilde{x}_0 = A_{t_1^{-1}}(x_0 - d_{B_1}(w_1)) + d_{B_1}(v_1).$$

We observe that $\psi^+_{2^s B_1}(\tilde{x}_0) \leqslant C\tilde{\psi}^+_{2^s B_1}(x_0)$ if d_3 and s are sufficiently large, where d_3 is as in the definition of $\tilde{\psi}^+_B$. (We may assume that s is sufficiently large.) We assume that $C_1 > d_3$, where C_1 is as in Proposition 2.2. By (2.11), (3.8) and Lemma 5.1 we then have

$$\begin{aligned}\mathrm{II} &\leqslant C2^{\gamma s}\sum_{B_1}\int \tilde{\psi}^+_{2^s B_1}(x_0)\chi_E(x_0)\,\mathrm{d}x_0\\ &\leqslant C2^{2\gamma s}\int \chi_E(x_0)\,\mathrm{d}x_0\\ &\leqslant C2^{2\gamma s}2^{-\epsilon_0 s^2}.\end{aligned}$$

Combining the results for $\ell = 0$ and $\ell = 1$, we have (5.6). □

Proof of (5.7). We consider the variables x_0, v, w in the range where $|\det(D_t(x_2))| \leqslant 2^{-\delta s}M_B$ and the integrand in (5.7) does not vanish for each $B \in \mathcal{B}_0$. We use results in the proof of Lemma 4.1. By (4.15) we have

$$\det(D_t(x_2)) = \langle A^*_{2^{k(B_2)+s}}L^*A_{2^{k(B_1)+s}}X, Y\rangle.$$

Note that $L^* = -L$. Therefore, the condition $|\det(D_t(x_2))| \leqslant 2^{-\delta s}M_B$ and (4.16) imply

$$|\langle W, Y\rangle| \leqslant C2^{-\delta s}(1 + |k(B_2) - k(B_1)|), \tag{5.8}$$

where $W = (c(k(B_2) - k(B_1))X_1 - X_2, X_1)$, $X = (X_1, X_2)$, $c = \log 2$.

First we assume that $|X_1| \geqslant C_1 2^{-\epsilon_1 s}$, $|k(B_2) - k(B_1)| \geqslant C_2 2^{\epsilon_2 s}$, $\epsilon_2 > \epsilon_1 > 0$. Let $Z = X_1 - X_2/(c(k(B_2) - k(B_1)))$. Then $|Z| \sim |X_1|$, if C_2 is sufficiently large. Therefore, by (5.8) we see that

$$|\langle(1, X_1(c(k(B_2) - k(B_1))Z)^{-1}), Y\rangle| \leqslant C|X_1|^{-1}2^{-\delta s} \leqslant C2^{-\delta s}2^{\epsilon_1 s}. \tag{5.9}$$

We note that

$$|X_1(c(k(B_2) - k(B_1))Z)^{-1}| \leqslant C2^{-\epsilon_2 s}.$$

Thus, (5.9) implies

$$|\langle e_1, Y\rangle| \leqslant C2^{-\delta s}2^{\epsilon_1 s} + C2^{-\epsilon_2 s}.$$

Therefore, recalling the definition of Y, we have

$$|\langle A^*_{t_2}P^*e_1, A_{2^{-k(B_2)-s}}(x_1 - d_{B_2}(v_2))\rangle| \leqslant C2^{-\delta s}2^{\epsilon_1 s} + C2^{-\epsilon_2 s}$$

and hence

$$|\langle A_{t_2}^* P^* e_1, A_{2^{-k(B_2)-s}}(x_1 - x_{B_2})\rangle| \leqslant C2^{-\delta s}2^{\epsilon_1 s} + C2^{-\epsilon_2 s} + C|A_{2^{-s}}(v_2)|$$
$$\leqslant C2^{-\delta_1 s} \tag{5.10}$$

for some $\delta_1 > 0$.

Next, we assume that $|X_1| \geqslant C_1 2^{-\epsilon_1 s}$, $|k(B_2) - k(B_1)| < C_2 2^{\epsilon_2 s}$. By (5.8) we then have

$$|\langle W, Y\rangle| \leqslant C2^{-\delta s}2^{\epsilon_2 s}.$$

We write $X = S + R$, where

$$S = t_1^{-1} P A_{t_1 t_2 2^{-k(B_1)-s}}(x_0 - x_{B_1}), \qquad R = -t_1^{-1} P A_{t_1 t_2 2^{-s}}(v_1)$$

and decompose W as $W = U + Q$, where

$$U = (c(k(B_2) - k(B_1))S_1 - S_2, S_1), \qquad Q = (c(k(B_2) - k(B_1))R_1 - R_2, R_1).$$

Here $S = (S_1, S_2)$, $R = (R_1, R_2)$. We note that $|R| \leqslant C2^{-\alpha' s}$ for any $\alpha' \in (0, \alpha)$. Therefore,

$$|\langle U, Y\rangle| \leqslant |\langle W, Y\rangle| + |\langle Q, Y\rangle| \leqslant C2^{-\delta s}2^{\epsilon_2 s} + C2^{-\alpha' s}2^{\epsilon_2 s}.$$

Also, if $|X_1| \geqslant C_1 2^{-\epsilon_1 s}$, $\epsilon_1 \in (0, \alpha)$ and C_1 is sufficiently large, we see that $|S_1| \geqslant C2^{-\epsilon_1 s}$ and hence $|U| \geqslant C2^{-\epsilon_1 s}$. Thus, if $U' = U/|U|$, we have

$$|\langle U', Y\rangle| \leqslant C2^{-\delta s}2^{\epsilon_2 s}2^{\epsilon_1 s} + C2^{-\alpha' s}2^{\epsilon_2 s}2^{\epsilon_1 s}.$$

As above, from this expression it follows that

$$|\langle A_{t_2}^* P^* U', A_{2^{-k(B_2)-s}}(x_1 - x_{B_2})\rangle| \leqslant C2^{-\delta_2 s} \tag{5.11}$$

for some $\delta_2 > 0$ with $\delta_2 > \epsilon_2$.

Let

$$V = \{x \in B(0, C')\colon |\langle A_{t_2}^* P^* e_1, x\rangle| \leqslant C2^{-\delta_1 s}\},$$
$$V_k = \{x \in B(0, C')\colon |\langle A_{t_2}^* P^* U_k', x\rangle| \leqslant C2^{-\delta_2 s}\}$$

for sufficiently large constants $C, C' > 0$, where $U_k = (c(k - k(B_1))S_1 - S_2, S_1)$, $U_k' = U_k/|U_k|$, $k \in \mathbb{Z}$.

By (5.10) and (5.11) we see that if $|X_1| \geqslant C_1 2^{-\epsilon_1 s}$, then

$$A_{2^{-k(B_2)-s}}(x_1 - x_{B_2}) \in S(B_1, x_0), \tag{5.12}$$

where

$$S(B_1, x_0) = V \cup \Biggl(\bigcup_{|k-k(B_1)| < C_2 2^{\epsilon_2 s}} V_k\Biggr).$$

We may assume that δ_1 and δ_2 are sufficiently small. By Lemma 5.2 we have

$$\left.\begin{aligned} V \subset \bigcup_j 2^{-1} S_j, \quad & \sum_j |S_j| \leqslant C2^{-\delta_1 s}, \\ V_k \subset \bigcup_j 2^{-1} S_j^k, \quad & \sum_j |S_j^k| \leqslant C2^{-\delta_2 s} \end{aligned}\right\} \tag{5.13}$$

for some balls S_j, S_j^k in $B(0,2C')$ with radius $2^{-\delta_0 s}$ for some $\delta_0 \in (0,1)$. In (5.3) we take this δ_0 and $C = 2C'$. By (5.12) and (5.13) we see that

$$\psi^+_{2^s B_2}(x_1) \leqslant C \sum_j \psi_{B_2,S_j}(x_1) + C \sum_{|k-k(B_1)|<C_2 2^{\epsilon_2 s}} \sum_j \psi_{B_2,S_j^k}(x_1).$$

Therefore, summing up in B_2 under the condition $A_{2^{-k(B_2)-s}}(x_1 - x_{B_2}) \in S(B_1,x_0)$ and $x_1 \in E^{\mathrm{c}}$, with the other variables (B_1, $x_0 \in \mathbb{R}^2$, $v_1, w_1 \in B(0,C)$) fixed, by (5.3) and (5.13) we have

$$\begin{aligned} \sum_{B_2} \psi^+_{2^s B_2}(x_1) &\leqslant C \sum_j U_{S_j}(x_1) + C \sum_{|k-k(B_1)|<C_2 2^{\epsilon_2 s}} \sum_j U_{S_j^k}(x_1) \\ &\leqslant C \sum_j s^3 2^{\gamma s} |S_j| + C \sum_{|k-k(B_1)|<C_2 2^{\epsilon_2 s}} \sum_j s^3 2^{\gamma s} |S_j^k| \\ &\leqslant C s^3 2^{\gamma s} 2^{-\delta_1 s} + C 2^{\epsilon_2 s} s^3 2^{\gamma s} 2^{-\delta_2 s} \\ &\leqslant C 2^{-\epsilon_3 s} 2^{\gamma s} \end{aligned} \tag{5.14}$$

for some $\epsilon_3 > 0$.

Let

$$R_B = \{(x_0,v,w) \colon |\det(D_t(x_2))| \leqslant 2^{-\delta s} M_B,\ |X_1| \geqslant C_1 2^{-\epsilon_1 s};\ v,w \in B(0,C)\},$$
$$R'_B = \{(x_0,v,w) \colon |\det(D_t(x_2))| \leqslant 2^{-\delta s} M_B,\ |X_1| < C_1 2^{-\epsilon_1 s};\ v,w \in B(0,C)\}.$$

To prove (5.7), we split the integral as follows:

$$\iiint_{|\det(D_t(x_2))| \leqslant 2^{-\delta s} M_B} \prod_{i=1}^2 (\psi^+_{2^s B_i}(x_{i-1}) \chi_{E^{\mathrm{c}}}(x_{i-1}) a^+_{B_i}(v_i) a^+_{B_i}(w_i)) \,\mathrm{d}x_0 \,\mathrm{d}v \,\mathrm{d}w$$
$$= \mathrm{I}_B + \mathrm{II}_B,$$

where

$$\mathrm{I}_B = \iiint_{R_B} \prod_{i=1}^2 (\psi^+_{2^s B_i}(x_{i-1}) \chi_{E^{\mathrm{c}}}(x_{i-1}) a^+_{B_i}(v_i) a^+_{B_i}(w_i)) \,\mathrm{d}x_0 \,\mathrm{d}v \,\mathrm{d}w,$$
$$\mathrm{II}_B = \iiint_{R'_B} \prod_{i=1}^2 (\psi^+_{2^s B_i}(x_{i-1}) \chi_{E^{\mathrm{c}}}(x_{i-1}) a^+_{B_i}(v_i) a^+_{B_i}(w_i)) \,\mathrm{d}x_0 \,\mathrm{d}v \,\mathrm{d}w.$$

From (3.8) and (5.12) it follows that

$$\mathrm{I}_B \leqslant C \iiint_{A_{2^{-k(B_2)-s}}(x_1-x_{B_2})\in S(B_1,x_0)} \prod_{i=1}^{2}(\psi^+_{2^sB_i}(x_{i-1})\chi_{E^c}(x_{i-1})) \times a^+_{B_1}(v_1)a^+_{B_1}(w_1)\,\mathrm{d}x_0\,\mathrm{d}v_1\,\mathrm{d}w_1.$$

Therefore, by (5.14), (3.8) and (2.8) we have

$$\begin{aligned}\sum_{B\in\mathcal{B}_0}\mathrm{I}_B &\leqslant C2^{-\epsilon_3 s}2^{\gamma s}\sum_{B_1\in\mathcal{B}}\int \psi^+_{2^sB_1}(x_0)\,\mathrm{d}x_0\\ &\leqslant C2^{-\epsilon_3 s}2^{\gamma s}\sum_{B_1\in\mathcal{B}}2^{\gamma s}|B_1|\\ &\leqslant C2^{-\epsilon_3 s}2^{2\gamma s}.\end{aligned} \tag{5.15}$$

To estimate II_B, by (3.8) we first see that

$$\mathrm{II}_B \leqslant C\iiint_{|X_1|<C_12^{-\epsilon_1 s}} \psi^+_{2^sB_1}(x_0)\psi^+_{2^sB_2}(x_1)\chi_{E^c}(x_1)a^+_{B_1}(v_1)a^+_{B_1}(w_1)\,\mathrm{d}x_0\,\mathrm{d}v_1\,\mathrm{d}w_1. \tag{5.16}$$

A change of variables implies that

$$\int_{|X_1|<C_12^{-\epsilon_1 s}} \psi^+_{2^sB_1}(x_0)\psi^+_{2^sB_2}(x_1)\chi_{E^c}(x_1)\,\mathrm{d}x_0 = t_1^{-\gamma}\int_{|\tilde{X}_1|<C_12^{-\epsilon_1 s}} \psi^+_{2^sB_1}(\tilde{x}_0)\psi^+_{2^sB_2}(x_0)\chi_{E^c}(x_0)\,\mathrm{d}x_0,$$

where $\tilde{x}_0$ is as in the proof of (5.6) and

$$\tilde{X}_1 = \langle e_1, t_1^{-1}A_{2^{-k(B_1)-s}}PA_{t_1t_2}(\tilde{x}_0 - d_{B_1}(v_1))\rangle.$$

We have $\psi^+_{2^sB_1}(\tilde{x}_0) \leqslant C\tilde{\psi}^+_{2^sB_1}(x_0)$ if d_3 and s are sufficiently large as in the proof of (5.6). Also, the condition $|\tilde{X}_1| < C_12^{-\epsilon_1 s}$ implies

$$|\langle a, A_{2^{-k(B_1)-s}}(x_0 - x_{B_1})\rangle| \leqslant C2^{-\epsilon_1 s} \tag{5.17}$$

for $\epsilon_1 \in (0,\alpha)$, where $a = A^*_{t_2}P^*e_1$. Therefore, by (5.16) and (3.8) we have

$$\mathrm{II}_B \leqslant C\int_{|\langle a, A_{2^{-k(B_1)-s}}(x_0-x_{B_1})\rangle|\leqslant C2^{-\epsilon_1 s}} \tilde{\psi}^+_{2^sB_1}(x_0)\chi_{E^c}(x_0)\psi^+_{2^sB_2}(x_0)\,\mathrm{d}x_0. \tag{5.18}$$

Arguing as in the proof of (5.14), if $x_0 \in E^c$, we see that

$$\sum_{B_1:\,|\langle a, A_{2^{-k(B_1)-s}}(x_0-x_{B_1})\rangle|\leqslant C2^{-\epsilon_1 s}} \tilde{\psi}^+_{2^sB_1}(x_0) \leqslant C2^{-\epsilon_4 s}2^{\gamma s} \tag{5.19}$$

for some $\epsilon_4 > 0$. Thus, from (5.18), (5.19) and (2.8) it follows that

$$\begin{aligned}\sum_{B\in\mathcal{B}_0} \mathrm{II}_B &\leqslant C2^{-\epsilon_4 s}2^{\gamma s} \sum_{B_2\in\mathcal{B}} \int \psi^+_{2^s B_2}(x_0)\,\mathrm{d}x_0 \\ &\leqslant C2^{-\epsilon_4 s}2^{\gamma s} \sum_{B_2\in\mathcal{B}} 2^{\gamma s}|B_2| \\ &\leqslant C2^{-\epsilon_4 s}2^{2\gamma s}. \end{aligned} \tag{5.20}$$

By (5.15) and (5.20) we have (5.7). □

6. Proof of Proposition 2.2 for P_2: proof of (4.3)

In this section we prove (4.3). By (3.10) it suffices to show that

$$\sum_{B\in\mathcal{B}_0} \left| \iiiint G_B(x_0)F_B(x_2)H_B\zeta_2(2^{\delta s}M_B^{-1}\det(D_t(x_2)))\,\mathrm{d}x_0\,\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}v \right| \leqslant C2^{-\epsilon s}\langle(2^{\gamma s}T^+)^2 1, 1\rangle.$$

Recalling the definition of T^+ in (3.9) and expanding $(T^+)^2$, we can see that this follows from

$$\left| \iiiint G_B(x_0)F_B(x_2)H_B\zeta_2(2^{\delta s}M_B^{-1}\det(D_t(x_2)))\,\mathrm{d}x_0\,\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}v \right| \leqslant C2^{-\epsilon s} \iiiint H_B^+(x_0,x_1,x_2,t,v,w)\,\mathrm{d}x_0\,\mathrm{d}w\,\mathrm{d}t\,\mathrm{d}v \tag{6.1}$$

for all $B \in \mathcal{B}_0$, where

$$H_B^+(x_0,x_1,x_2,t,v,w) = \prod_{i=1}^{2}(\psi^+_{2^s B_i}(x_{i-1})a^+_{B_i}(v_i)\varphi^+(t_i)a^+_{B_i}(w_i)\psi^+_{2^s B_i}(x_i)).$$

If we fix all the variables but y, t, then (6.1) follows from the estimate

$$\left| \iint F_B(x_2)a_{B_1}(y)L(y,t)\,\mathrm{d}y\,\mathrm{d}t \right| \leqslant C2^{-\epsilon s} \iint a^+_{B_1}(y)L^+(y,t)\,\mathrm{d}y\,\mathrm{d}t, \tag{6.2}$$

which is uniform in the fixed variables, where

$$L(y,t) = \prod_{i=1}^{2}(\psi_{2^s B_i}(x_{i-1})\psi_{2^s B_i}(x_i)\tilde{\varphi}(t_i))\zeta_2(2^{\delta s}M_B^{-1}\det(D_t(x_2))), \tag{6.3}$$

$$L^+(y,t) = \prod_{i=1}^{2}(\psi^+_{2^s B_i}(x_{i-1})\psi^+_{2^s B_i}(x_i)\varphi^+(t_i)). \tag{6.4}$$

To prove (6.2), by (3.7) it suffices to show

$$\left| \iint F_B(x_2)L(y,t)\partial_{y_i}a^i_{B_1}(y)\,\mathrm{d}y\,\mathrm{d}t \right| \leqslant C2^{-\epsilon s} \iint a^+_{B_1}(y)L^+(y,t)\,\mathrm{d}y\,\mathrm{d}t \tag{6.5}$$

for $i = 1, 2$. Fix i. Applying integration by parts, we can see that the left-hand side of (6.5) is majorized by

$$\left| \iint F_B(x_2) a^i_{B_1}(y) \partial_{y_i} L(y,t)\,\mathrm{d}y\,\mathrm{d}t \right| + \left| \iint a^i_{B_1}(y) L(y,t) \partial_{y_i} F_B(x_2)\,\mathrm{d}y\,\mathrm{d}t \right|. \tag{6.6}$$

To estimate this, we need the following.

Lemma 6.1. *Let L and L^+ be as in (6.3) and (6.4), respectively. Then we have*

$$|L(y,t)| + s^{-1} 2^{\alpha s} |\partial_{y_j} L(y,t)| + |\partial_{t_k} L(y,t)| \leqslant C 2^{\delta s} L^+(y,t)$$

for all y, t and $j, k = 1, 2$.

Proof. We note that

$$s^{-1} 2^{\alpha s} |\partial_{y_j} \zeta_2(2^{\delta s} M_B^{-1} \det(D_t(x_2)))| + |\partial_{t_k} \zeta_2(2^{\delta s} M_B^{-1} \det(D_t(x_2)))| \leqslant C 2^{\delta s} \tag{6.7}$$

on the support of L. This follows from (4.4) and the chain rule. The estimates (4.6) and (6.7) imply the conclusion of Lemma 6.1. □

By Lemma 6.1, we can estimate the first term of (6.6) as follows:

$$\left| \iint F_B(x_2) a^i_{B_1}(y) \partial_{y_i} L(y,t)\,\mathrm{d}y\,\mathrm{d}t \right| \leqslant C s 2^{(\delta-\alpha)s} \iint a^+_{B_1}(y) L^+(y,t)\,\mathrm{d}y\,\mathrm{d}t. \tag{6.8}$$

An estimate needed for the second term of (6.6) follows if we prove that

$$\left| \int L(y,t) \partial_{y_i} F_B(x_2)\,\mathrm{d}t \right| \leqslant C 2^{-\epsilon s} \int L^+(y,t)\,\mathrm{d}t \tag{6.9}$$

uniformly in y. To prove (6.9), we use the following [**15**].

Lemma 6.2. *Suppose that* $\det D_t(x_2) \neq 0$. *We then have the equality*

$$\partial_{y_i} F_B(x_2) = \langle \nabla_t (F_B(x_2)(1,1)), D_t(x_2)^{-1} (\partial_{y_i} x_2) \rangle,$$

where $\nabla_t(g_1, g_2) = (\partial_{t_1} g_1, \partial_{t_2} g_2)$ and $F_B(x_2)(1,1) = (F_B(x_2), F_B(x_2))$.

Fix y. By Lemma 6.2, we can write the left-hand side of (6.9) as

$$\left| \int L(y,t) \langle \nabla_t (F_B(x_2)(1,1)), D_t(x_2)^{-1} (\partial_{y_i} x_2) \rangle\,\mathrm{d}t \right|.$$

Integration by parts implies that this is equal to

$$\left| \int F_B(x_2) \langle (1,1), \nabla_t (L(y,t) D_t(x_2)^{-1} (\partial_{y_i} x_2)) \rangle\,\mathrm{d}t \right|.$$

Therefore, by Lemma 6.1, to prove (6.9) it suffices to show that

$$|D_t(x_2)^{-1} (\partial_{y_i} x_2)| + |\nabla_t (D_t(x_2)^{-1} (\partial_{y_i} x_2))| \leqslant C 2^{-\epsilon s} 2^{-\delta s} \tag{6.10}$$

on the support of $L(y,t)$. By Cramer's rule, (6.10) is a consequence of the estimates

$$\left|\frac{\det(D_{y_i,t_j}(x_2))}{\det D_t(x_2)}\right| + \left|\partial_{t_k}\frac{\det(D_{y_i,t_j}(x_2))}{\det D_t(x_2)}\right| \leqslant Cs2^{-\alpha s}2^{2\delta s}, \quad j,k=1,2,$$

which follows from (4.4), (4.5) and the estimate $|\det D_t(x_2)| \geqslant C2^{-\delta s}M_B$ on the support of L. This proves (6.10) with $\epsilon = \alpha' - 3\delta$ for any $\alpha' \in (0,\alpha)$. Thus, we have (6.9) with $\epsilon = \alpha' - 3\delta$. Combining this with (6.8), we have (6.5) with $\epsilon = \alpha' - 3\delta$, choosing δ to be sufficiently small. This completes the proof of (4.3).

7. Proof of Proposition 2.2 for P_3

In this section we consider the case $P = P_3$, where P_3 is as in (1.2). Then $A_t = t^\alpha U_t$, where

$$U_t = \begin{pmatrix} \cos(\beta\log t) & \sin(\beta\log t) \\ -\sin(\beta\log t) & \cos(\beta\log t)\end{pmatrix}.$$

Let

$$M_B = 2^{\alpha(k(B_1)+s)}2^{\alpha(k(B_2)+s)} \tag{7.1}$$

for $B=(B_1,B_2)\in\mathcal{B}^2$. Let $D_t(x_2)$, $D_{y_i,t_j}(x_2)$, for $i,j=1,2$, be as in § 4 with $P=P_3$. The following lemma can then be proved in the same way as Lemma 4.1 by noting $U_t \in \mathrm{SO}(2)$.

Lemma 7.1. *Let M_B be as in (7.1) and let $B\in\mathcal{B}_0$, where $\mathcal{B}_0$ is as in (3.14). Let $t_\ell \in [C^{-1},C]$, $v_\ell \in B(0,C)$, $x_{\ell-1}\in \mathrm{supp}(\psi^+_{2^sB_\ell})$, $\ell=1,2$. Then the following estimates hold:*

$$|\det(D_t(x_2))| + 2^{\alpha s}|\partial_{y_i}\det(D_t(x_2))| + |\partial_{t_j}\det(D_t(x_2))| \leqslant CM_B, \tag{7.2}$$

$$2^{\alpha s}|\det(D_{y_i,t_j}(x_2))| + 2^{\alpha s}|\partial_{t_k}\det(D_{y_i,t_j}(x_2))| \leqslant CM_B \tag{7.3}$$

for $i,j,k=1,2$, and

$$|\psi_{2^sB_\ell}(x_{\ell'})| + 2^{\alpha s}|\partial_{y_i}\psi_{2^sB_\ell}(x_{\ell'})| + |\partial_{t_j}\psi_{2^sB_\ell}(x_{\ell'})| \leqslant C\psi^+_{2^sB_\ell}(x_{\ell'}) \tag{7.4}$$

for $i,j=1,2$, $0\leqslant \ell'\leqslant \ell$, $\ell=1,2$.

To prove Theorem 1.2 for P_3, it suffices to prove Proposition 2.2 for P_3. So, we have to prove estimates analogous to (4.2) and (4.3) in the case of P_3 with M_B in (7.1). To prove an analogue of (4.2), we show analogues of (5.6) and (5.7). An analogue of (5.6) can be shown in the same way as in the case of P_2. To prove an analogue of (5.7), by (4.15) for P_3 we note that

$$\begin{aligned}\det(D_t(x_2)) &= \langle A^*_{2^{k(B_2)+s}}L^*A_{2^{k(B_1)+s}}X,Y\rangle \\ &= 2^{(k(B_1)+s)\alpha}2^{(k(B_2)+s)\alpha}\langle U_{2^{-k(B_2)-s}}L^*U_{2^{k(B_1)+s}}X,Y\rangle,\end{aligned}$$

where X and Y are as in (4.15) with $P = P_3$. Suppose that $\beta = 2\pi k/\log 2$ for some $k \in \mathbb{Z}$. Then U_{2^j} is the identity matrix for all $j \in \mathbb{Z}$. So we have

$$\det(D_t(x_2)) = 2^{(k(B_1)+s)\alpha} 2^{(k(B_2)+s)\alpha} \langle L^* X, Y\rangle.$$

Therefore, if $|\det(D_t(x_2))| \leqslant 2^{-\delta s} M_B$ and the integrand in (5.7) does not vanish, noting that $L^* = -L$, we see that $|\langle LX, Y\rangle| \leqslant C2^{-\delta s}$. If

$$S = t_1^{-1} P A_{t_1 t_2 2^{-k(B_1)-s}}(x_0 - x_{B_1})$$

as in the proof of (5.7), this implies $|\langle LS, Y\rangle| \leqslant C2^{-\delta s}$ for $\delta \in (0, \alpha)$. Also, from the inequality $|X_1| \geqslant C_1 2^{-\epsilon_1 s}$, $\epsilon_1 \in (0, \alpha)$, it follows that $|S_1| \geqslant C2^{-\epsilon_1 s}$ if C_1 is sufficiently large. It follows that

$$|\langle LS/|LS|, Y\rangle| \leqslant C2^{-\delta s} 2^{\epsilon_1 s}.$$

This estimate along with the definition of Y implies

$$|\langle A_{t_2}^* P^*(LS/|LS|), A_{2^{-k(B_2)-s}}(x_1 - d_{B_2}(v_2))\rangle| \leqslant C2^{-\delta s} 2^{\epsilon_1 s}.$$

It follows that

$$\begin{aligned} |\langle A_{t_2}^* P^*(LS/|LS|), A_{2^{-k(B_2)-s}}(x_1 - x_{B_2})\rangle| &\leqslant C2^{-\delta s} 2^{\epsilon_1 s} + C|A_{2^{-s}}(v_2)| \\ &\leqslant C2^{-\delta_1 s} \end{aligned} \tag{7.5}$$

for some $\delta_1 > 0$, if $|X_1| \geqslant C_1 2^{-\epsilon_1 s}$. Therefore, if we fix the variables except for B_2, then $A_{2^{-k(B_2)-s}}(x_1 - x_{B_2})$ lies in a $C2^{-\delta_1 s}$ neighbourhood of a line. Also, if $|X_1| < C_1 2^{-\epsilon_1 s}$, results similar to those in § 5 hold (see, for example, (5.17)). Thus, an analogue of (5.7) in the case of P_3 can be proved as in § 5 (see (5.15), (5.20)).

To prove an analogue of (4.3) we first note the following.

Lemma 7.2. *Let L and L^+ be defined as in (6.3) and (6.4), respectively, with everything adapted for the present case. Then we have the pointwise estimates*

$$|L(y,t)| + 2^{\alpha s}|\partial_{y_j} L(y,t)| + |\partial_{t_k} L(y,t)| \leqslant C2^{\delta s} L^+(y,t)$$

for $j, k = 1, 2$.

We can prove this by using Lemma 7.1, in the same way as we proved Lemma 6.1 by applying Lemma 4.1.

By Lemmas 7.1 and 7.2 we can prove an analogue of the estimate (6.5) for the present situation, which will prove an analogue of (4.3) as in § 6.

We have just proved Theorem 1.2 for P_3 assuming $\beta = 2\pi k/\log 2$ for some $k \in \mathbb{Z}$. Now we remove the restriction on β. Let $D_t = A_{t^\lambda}$, $\lambda > 0$, and $r_D(x) = r(x)^{1/\lambda}$. Then, $D_t = \exp((\lambda \log t) P_3)$ and $r_D(D_t x) = t r_D(x)$, $K(D_t x) = t^{-\lambda\gamma} K(x)$ for $x \in \mathbb{R}^2 \setminus \{0\}$, $t > 0$. Also, we can easily see that D_t, r_D and K satisfy all the conditions in Theorem 1.2 assumed for A_t, r and K. Furthermore, if we choose λ such that $\lambda\beta = 2\pi k/\log 2$ for some $k \in \mathbb{Z}$, then the proof of Theorem 1.2 given above under the restriction of β applies to the proof of Theorem 1.2 for D_t, r_D and K. This proves Theorem 1.2 for a general P_3.

8. Reduction to the Jordan canonical forms

We choose a non-singular real matrix Q such that $Q^{-1}PQ$ is one of the three matrices in (1.2). Let $R = Q^{-1}PQ$. Then $Q^{-1}A_tQ = t^R$. Put $D_t = t^R$. Set $K_1(x) = (\det Q)K(Qx)$. Then $K_1(D_tx) = t^{-\gamma}K_1(x)$ for $x \in \mathbb{R}^2 \setminus \{0\}$, $t > 0$. Put $r_1(x) = r(Qx)$. Then $r_1(D_tx) = tr_1(x)$ and $r_1(x) = 1$ if and only if $\langle Q^*BQx, x\rangle = 1$, where B is as in statement (iii) of § 1. We note that Q^*BQ is positive and symmetric. Also, we have

$$\int_{a<r_1(x)<b} K_1(x)\,\mathrm{d}x = \int_{a<r(x)<b} K(x)\,\mathrm{d}x = 0 \quad \text{for all } a, b \text{ with } 0 < a < b.$$

Furthermore, if $E_0 = \{x \in \mathbb{R}^2 \colon 1 \leqslant r_1(x) \leqslant 2\}$, then $K_1(x)\chi_{E_0}(x) \in L\log L(\mathbb{R}^2)$.

Define

$$T_1f(x) = \text{p.v.}\int f(y)K_1(x-y)\,\mathrm{d}y.$$

Theorem B, Remark 1.1 and what we have already proved then imply the weak-type $(1,1)$ estimate for T_1:

$$|\{x \in \mathbb{R}^2 \colon |T_1f(x)| > \lambda\}| \leqslant C\lambda^{-1}\|f\|_1, \tag{8.1}$$

since K_1, D_t and r_1 satisfy all the requirements needed in the proof. We note that $T_1f(x) = Tf_Q(Qx)$, where $f_Q(x) = f(Q^{-1}x)$. Using this and changing variables in (8.1), we can see that T is of weak-type $(1,1)$.

Appendix A

A.1. Proof of (2.7) from Proposition 2.1

First, by dilation invariance we may assume that $c \leqslant \sum|B| \leqslant 1$ in (2.7) for some constant $c > 0$. For $s > C$, we decompose K_0 as $K_0 = H^{(s)} + L^{(s)}$ with $L^{(s)} = K_0\chi_{\{|K_0|\leqslant 2^{\epsilon s/2}\}}$, where ϵ is as in Proposition 2.1. Then we have to prove

$$\left|\left\{\sum_{s>C}\left|\sum_B \psi_{2^sB}(b_B * S_{k(B)+s}H^{(s)})\right| > \tfrac{1}{6}\right\}\right| \leqslant C_1, \tag{A 1}$$

$$\left|\left\{\sum_{s>C}\left|\sum_B \psi_{2^sB}(b_B * S_{k(B)+s}L^{(s)})\right| > \tfrac{1}{6}\right\}\right| \leqslant C_1 \tag{A 2}$$

for some positive constant C_1. The estimates (A 1) and (A 2) imply (2.7). The estimate (A 1) follows from

$$\left\|\sum_{s>C}\left|\sum_B \psi_{2^sB}(b_B * S_{k(B)+s}H^{(s)})\right|\right\|_1 \leqslant C \tag{A 3}$$

by Chebyshev's inequality. To see this, we note that the estimates (2.1) and (2.5) imply

$$\|\psi_{2^sB}(b_B * S_{k(B)+s}H^{(s)})\|_1 \leqslant C|B|\,\|H^{(s)}\|_1. \tag{A 4}$$

Since

$$\sum_{s>C}\|H^{(s)}\|_1 \leqslant C\|K_0\|_{L\log L} = C,$$

(2.8) and (A 4) imply (A 3).

To prove (A 2) we note that $|\bigcup_{s>C} E_s| \leqslant C$. Thus, by Chebyshev's inequality it suffices to show that

$$\left\| \sum_{s>C} \left| \sum_{B} \psi_{2^s B}(b_B * S_{k(B)+s} L^{(s)}) \right| \right\|_{L^p(F^c)} \leqslant C, \tag{A 5}$$

where $F = \bigcup_{s>C} E_s$. The estimate (A 5) follows from

$$\left\| \sum_{B} \psi_{2^s B}(b_B * S_{k(B)+s} L^{(s)}) \right\|_{L^p(E_s^c)} \leqslant C 2^{-\epsilon s/2} \tag{A 6}$$

by the triangle inequality. We can prove (A 6) by Proposition 2.1 with $f_B = L^{(s)}$ for all B, since

$$\left(\sum_{B} |B| \, \|L^{(s)}\|_2^2 \right)^{1/2} \leqslant C \|L^{(s)}\|_2 \leqslant C 2^{\epsilon s/2}.$$

A.2. Proof of Lemma 2.3

We prove

$$\left\| \sum_{B \in \mathcal{B}} \psi_{B,S} \right\|_1 \leqslant C 2^{\gamma s} |S|, \tag{A 7}$$

$$\left\| \sum_{B \in \mathcal{B}} \psi_{B,S} \right\|_{\mathrm{BMO}} \leqslant C s 2^{\gamma s} |S|, \tag{A 8}$$

where BMO is the space defined by using the balls with respect to the function r. The estimates (A 7) and (A 8) imply the conclusion of Lemma 2.3, since we have

$$|\{|f| > \lambda\}| \leqslant C \exp(-A\lambda/\|f\|_{\mathrm{BMO}}) \|f\|_1/\lambda$$

for some $A > 0$, which follows from the John–Nirenberg inequality [**8**].

Proof of (A 7) is straightforward:

$$\left\| \sum_{B \in \mathcal{B}} \psi_{B,S} \right\|_1 \leqslant \sum_{B \in \mathcal{B}} \|\psi_{B,S}\|_1 \leqslant C \sum_{B \in \mathcal{B}} 2^{\gamma s} |S| \, |B| \leqslant C 2^{\gamma s} |S|,$$

where the last inequality follows from (2.8).

To prove (A 8), it suffices to show that

$$\sup_R \sum_B \mathcal{O}_R(\psi_{B,S}) \leqslant C s 2^{\gamma s} |S|,$$

where

$$\mathcal{O}_R(f) = |R|^{-1} \int_R |f - f_R|, \quad f_R = |R|^{-1} \int_R f.$$

Fix a ball $R = B(x_R, u)$. Take $i \in \mathbb{Z}$ such that $2^i \leqslant u < 2^{i+1}$.

Case 1 ($\boldsymbol{i \geqslant k(B)+s}$). If $\mathcal{O}_R(\psi_{B,S}) \neq 0$, then $R \cap C2^s B \neq \emptyset$ for some $C > 0$, and hence

$$r(x_B - x_R) \leqslant C(u + 2^{k(B)+s}) \leqslant Cu,$$

which implies $B \subset CR$. Therefore, since $\mathcal{O}_R(\psi_{B,S}) \leqslant C|R|^{-1}2^{\gamma s}|B|\,|S|$, we have

$$\sum_{B:\, i \geqslant k(B)+s} \mathcal{O}_R(\psi_{B,S}) \leqslant C \sum_{B \subset CR} |R|^{-1}2^{\gamma s}|S|\,|B| \leqslant C2^{\gamma s}|S|.$$

Case 2 ($\boldsymbol{k(B)+s-\delta s < i < k(B)+s}$). If $\mathcal{O}_R(\psi_{B,S}) \neq 0$, there exists x such that $r(x - x_R) < u$ and $r(A_{2^{-k(B)-s}}(x - x_B) - x_S) \leqslant C2^{-\delta s}$. Thus,

$$\begin{aligned} r(x_B + A_{2^{k(B)+s}}x_S - x_R) &\leqslant C_0 r(x - x_R) + C_0 r(x_B + A_{2^{k(B)+s}}x_S - x) \\ &\leqslant C(u + 2^{k(B)+s-\delta s}) \\ &\leqslant Cu, \end{aligned}$$

where C_0 is as in statement (ii) of § 1. It follows that $B + A_{2^{k(B)+s}}x_S \subset CR$, where $B + a = \{x + a\colon x \in B\}$, $a \in \mathbb{R}^n$. For $j \in \mathbb{Z}$, define a family of disjoint balls

$$\mathcal{I}_j = \{B \in \mathcal{B}\colon \mathcal{O}_R(\psi_{B,S}) \neq 0,\ k(B) = j\}.$$

Then

$$\begin{aligned} \sum_{B:\, k(B)+s-\delta s<i<k(B)+s} \mathcal{O}_R(\psi_{B,S}) &\leqslant C \sum_{i-s<j<i-s+\delta s}\ \sum_{B \in \mathcal{I}_j} |R|^{-1}2^{\gamma s}|B|\,|S| \\ &\leqslant C \sum_{i-s<j<i-s+\delta s} |R|^{-1}2^{\gamma s}|CR - A_{2^{j+s}}x_S|\,|S| \\ &\leqslant C\delta s 2^{\gamma s}|S|. \end{aligned}$$

Case 3 ($\boldsymbol{k(B) \leqslant i \leqslant k(B)+s-\delta s}$). As in Case 2 we have

$$r(x_B + A_{2^{k(B)+s}}x_S - x_R) \leqslant C2^{k(B)+s-\delta s},$$

if $\mathcal{O}_R(\psi_{B,S}) \neq 0$. This implies

$$B + A_{2^{k(B)+s}}x_S \subset B(x_R, C2^{k(B)+s-\delta s}).$$

Thus, we have

$$\mathrm{card}(\mathcal{I}_j)2^{\gamma j} \leqslant C2^{\gamma(j+s-\delta s)}$$

if $j \leqslant i \leqslant j + s - \delta s$, where $\mathcal{I}_j$ is as above. Since $\mathcal{O}_R(\psi_{B,S}) \leqslant C$, it follows that

$$\begin{aligned} \sum_{B:\, k(B) \leqslant i \leqslant k(B)+s-\delta s} \mathcal{O}_R(\psi_{B,S}) &\leqslant \sum_{i-s+\delta s \leqslant j \leqslant i}\ \sum_{B \in \mathcal{I}_j} \mathcal{O}_R(\psi_{B,S}) \\ &\leqslant C \sum_{i-s+\delta s \leqslant j \leqslant i} \mathrm{card}(\mathcal{I}_j) \\ &\leqslant Cs2^{\gamma s}|S|. \end{aligned}$$

Case 4 ($i < k(B)$). As in Case 3 we have $\mathrm{card}(\mathcal{I}_j) \leqslant C2^{\gamma s}|S|$ for $j > i$. Now we have

$$\mathcal{O}_R(\psi_{B,S}) \leqslant |R|^{-2} \iint_{R\times R} |\psi_{B,S}(x) - \psi_{B,S}(y)| \, \mathrm{d}x \, \mathrm{d}y.$$

Note that

$$|\psi_{B,S}(x) - \psi_{B,S}(y)| \leqslant C|A_{u_S^{-1}2^{-k(B)-s}}(x-y)| \leqslant C2^{(\delta s - k(B) - s + i)/\beta_1}$$

for $x, y \in R$, where β_1 is as in statement (v) of § 1. Therefore,

$$\begin{aligned}\sum_{B\colon k(B)>i} \mathcal{O}_R(\psi_{B,S}) &\leqslant \sum_{j>i} \sum_{B\in\mathcal{I}_j} \mathcal{O}_R(\psi_{B,S}) \\ &\leqslant C \sum_{j>i} \mathrm{card}(\mathcal{I}_j) 2^{(\delta s - j - s + i)/\beta_1} \\ &\leqslant C2^{\gamma s}|S|.\end{aligned}$$

Combining results in Cases 1–4, we have (A 8).

A.3. Proof of Proposition 2.1 from Proposition 2.2 and Lemma 2.3

For $B \in \mathcal{B}$ and a constant $D > 0$, let

$$h(B) = \mathrm{card}(\{B' \in \mathcal{B}\colon C_0 D 2^s B \subset C_0 D 2^s B'\}),$$

where $\mathcal{B}$ is as in Proposition 2.1 and C_0 is as in statement (ii) of § 1. Note that

$$\left| \bigcup_{h(B)\geqslant s^3 2^{\gamma s}} D2^s B \right| \leqslant \left| \left\{ \sum_{B\in\mathcal{B}} \chi_{C_0 D 2^s B} \geqslant s^3 2^{\gamma s} \right\} \right| \leqslant C2^{-\epsilon s^2}$$

for some $\epsilon > 0$, where the last inequality follows from Lemma 2.3 with $S = B(0, 2C_0 D)$. We can put $E_s = \bigcup_{h(B)\geqslant s^3 2^{\gamma s}} D2^s B$ in Proposition 2.1.

Let

$$\mathcal{B}_\ell = \{B \in \mathcal{B}\colon \ell 2^{\gamma s} \leqslant h(B) < (\ell+1)2^{\gamma s}\}$$

for $\ell = 0, 1, \dots, s^3 - 1$. We show that $\mathcal{B}_\ell$ satisfies (2.11) in place of $\mathcal{B}$ if D is large enough. Then, if we also take D satisfying $D > d_1$, where d_1 is as in the definition of ψ_B, by the definition of E_s the estimate (2.10) follows from s^3 applications of (2.12) and the triangle inequality.

Let

$$\mathcal{B}^x = \{B \in \mathcal{B}_\ell\colon x \in D2^{s-1}B\}$$

for an arbitrary x and the constant D satisfying $D/2 \geqslant C_1$, where C_1 is as in Proposition 2.2. We show that $\mathrm{card}(\mathcal{B}^x) \leqslant C2^{\gamma s}$. We may assume that $\mathcal{B}^x \neq \emptyset$. Let B_0 have the minimal radius 2^{j_0} in $\mathcal{B}^x$ and let B_1 have the maximal radius 2^{j_1} in $\mathcal{B}^x$. For $j_0 \leqslant j \leqslant j_1$, we note that

$$\mathrm{card}(\{B \in \mathcal{B}^x\colon k(B) = j\}) \leqslant C2^{\gamma s}. \tag{A 9}$$

Take $m \in \mathbb{Z}$ such that $2^{m-1} < C_0^2 \leqslant 2^m$. Suppose that $j_1 > j_0 + 2 + m$. Then we have

$$h(B_0) \geqslant h(B_1) + \mathrm{card}(\{B \in \mathcal{B}^x \colon j_0 + 2 + m \leqslant k(B) < j_1\}). \tag{A 10}$$

To show this, let $x \in D2^{s-1}B_0 \cap D2^{s-1}B$, $B = B(z, 2^j)$, $j_0 + 2 + m \leqslant j < j_1$, $B_0 = B(w, 2^{j_0})$. If $y \in C_0 D2^s B_0$, then

$$\begin{aligned} r(y-z) &\leqslant C_0^2 r(y-w) + C_0^2 r(w-x) + C_0 r(x-z) \\ &\leqslant C_0^3 D 2^{j_0+s} + C_0^2 D 2^{j_0+s-1} + C_0 D 2^{j+s-1} \\ &\leqslant C_0^3 D 2^{j_0+s+1} + C_0 D 2^{j+s-1} \\ &\leqslant C_0 D 2^{j+s}, \end{aligned}$$

which implies $C_0 D2^s B_0 \subset C_0 D2^s B$. Similarly, this argument implies $C_0 D2^s B_0 \subset C_0 D2^s B_1$. Thus, if $C_0 D2^s B_1 \subset C_0 D2^s B'$, then

$$C_0 D2^s B_0 \subset C_0 D2^s B_1 \subset C_0 D2^s B'.$$

From these results (A 10) follows. By (A 10) we have

$$\mathrm{card}(\{B \in \mathcal{B}^x \colon j_0 + 2 + m \leqslant k(B) < j_1\}) \leqslant h(B_0) - h(B_1) \leqslant 2^{\gamma s}.$$

Combining this with (A 9), we have $\mathrm{card}(\mathcal{B}^x) \leqslant C 2^{\gamma s}$ as claimed.

A.4. Proof of (3.10)

By interpolation and duality, to prove (3.10) it suffices to show the claim with $q = \infty$. To achieve this, by the positivity of the operator we may assume that F is identically equal to 1. Therefore, we must show that

$$\left\| 2^{-\gamma s} \sum_{B \in \mathcal{B}} \psi^+_{2^s B} T^+_B \psi^+_{2^s B} \right\|_p \leqslant C.$$

Since we are assuming $C_1 \geqslant d_2$, where C_1 is as in (2.11) and d_2 is as in the definition of ψ^+_B, by (2.11) and Hölder's inequality we have

$$2^{-\gamma s} \sum_{B \in \mathcal{B}} \psi^+_{2^s B} T^+_B \psi^+_{2^s B} \leqslant C 2^{-\gamma s/p} \left(\sum_{B \in \mathcal{B}} (T^+_B \psi^+_{2^s B})^p \right)^{1/p}. \tag{A 11}$$

Since $\|T^+_B F\|_p \leqslant C \|F\|_p$ uniformly in B by (3.8) and Minkowski's inequality, using the pointwise estimate (A 11), we see that

$$\begin{aligned} \left\| 2^{-\gamma s} \sum_{B \in \mathcal{B}} \psi^+_{2^s B} T^+_B \psi^+_{2^s B} \right\|_p &\leqslant C 2^{-\gamma s/p} \left(\sum_{B \in \mathcal{B}} \|T^+_B \psi^+_{2^s B}\|_p^p \right)^{1/p} \\ &\leqslant C 2^{-\gamma s/p} \left(\sum_{B \in \mathcal{B}} \|\psi^+_{2^s B}\|_p^p \right)^{1/p} \\ &\leqslant C 2^{-\gamma s/p} \left(\sum_{B \in \mathcal{B}} 2^{s\gamma} |B| \right)^{1/p} \\ &\leqslant C, \end{aligned}$$

where the last inequality follows from (2.8). This completes the proof of (3.10).

References

1. A. P. Calderón, Inequalities for the maximal function relative to a metric, *Studia Math.* **57** (1976), 297–306.
2. A. P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution, *Adv. Math.* **16** (1975), 1–64.
3. M. Christ, Weak-type $(1,1)$ bounds for rough operators, *Annals Math.* **128** (1988), 19–42.
4. M. Christ and J. L. Rubio de Francia, Weak-type $(1,1)$ bounds for rough operators, II, *Invent. Math.* **93** (1988), 225–237.
5. D. Fan and S. Sato, Weighted weak-type $(1,1)$ estimates for singular integrals and Littlewood–Paley functions, *Studia Math.* **163** (2004), 119–136.
6. S. Hofmann, Weak $(1,1)$ boundedness of singular integrals with nonsmooth kernel, *Proc. Am. Math. Soc.* **103** (1988), 260–264.
7. S. Hofmann, Weighted weak-type $(1,1)$ inequalities for rough operators, *Proc. Am. Math. Soc.* **107** (1989), 423–435.
8. R. Long, Z. Shen and Y. Yang, Weighted inequalities concerning maximal operator and #-operator on spaces of homogeneous type, *Approx. Theory Applicat.* **1** (1985), 53–72.
9. N. Rivière, Singular integrals and multiplier operators, *Ark. Mat.* **9** (1971), 243–278.
10. S. Sato, Some weighted weak type estimates for rough operators, *Math. Nachr.* **87** (1997), 211–240.
11. S. Sato, Estimates for singular integrals along surfaces of revolution, *J. Austral. Math. Soc.* **86** (2009), 413–430.
12. A. Seeger, Singular integral operators with rough convolution kernels, *J. Am. Math. Soc.* **9** (1996), 95–105.
13. A. Seeger and T. Tao, Sharp Lorentz space estimates for rough operators, *Math. Annalen* **320** (2001), 381–415.
14. E. M. Stein and S. Wainger, Problems in harmonic analysis related to curvature, *Bull. Am. Math. Soc.* **84** (1978), 1239–1295.
15. T. Tao, The weak-type $(1,1)$ of $L \log L$ homogeneous convolution operator, *Indiana Univ. Math. J.* **48** (1999), 1547–1584.
16. A. Vargas, Weighted weak-type $(1,1)$ bounds for rough operators, *J. Lond. Math. Soc. (2)* **54** (1996), 297–310.