A band-limited signal extrapolation algorithm using waveform synthesis digital filter

A BAND-LIMITED SIGNAL EXTRAPOLATION ALGORITHM USING WAVEFORM SYNTHESIS DIGITAL FILTER

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Abstract: An extrapolation algorithm for discrete band-limited signals is proposed. The band-limited condition is satisfied by using an ideal band-limited signal as an excited signal. A waveform synthesis digital filter is employed to extrapolate fractionally observed data, using the band-limited excited signal. Therefore, the proposed method reduces the extrapolation problem to a digital filter estimation problem. Digital filter coefficients are obtained by solving linear equations, which are formulated so as to minimize the mean square error between a part of the extrapolated signal and the fractionally observed data. The modified Gram-Schmidt (MGS) procedure is applied to finding the least square solution of the filter coefficients, due to its superior numerical stability. The recursive version of the MGS procedure is suited to real time hardware implememtation.

Through numerical examples, the proposed algorithm is recognized to have good performance in extrapolating the fractional data, observed in a single as well as multiple time windows. This concept is applicable not only to the signal extrapolation but also to the spectral estimation.

I. INTRODUCTION

One of the essential signal reconstruction problems is how to extrapolate a discrete band-limited signal x(n) for any n in terms of a finite segment of x(n). Several extrapolation techniques have been proposed, which can be classified into three categories: 1) the analytical continuation technique, which is mainly based on the Taylor series expansion; 2) the prolate spheroidal wave function technique, based on a special set of basis functions, which are orthogonal over the observation interval as well as over an infinite interval [1]; and 3) the iterative algorithm [2]-[4].In general, methods 1) and 2) are hardly implemented numerically. The iterative algorithm consists of sequentially applying a set of signal operations to generate an approximation sequence with convergence to the desired signal. However, the convergence rate for this algorithm is relatively slow and a large number of iterations is required. Along this direction, Sabri and Steenaart proposed a matrix method [5]. Cadzow also developed a convenient one-step extrapolation procedure [6].

This paper presents a new extrapolation algorithm, based on another point of view. The band-limited signal condition is satisfied by using the ideally band-limited signal as an excited signal. A waveform synthesis digital filter is employed to extrapolate the fractionally observed data, using the band-limited exciting signal. Digital filter coefficients are obtained by solving linear equations. Hardware implementation is briefly discussed, using the recursive version of the Gram-Schmidt procedure. Several numerical examples are given to evaluate the proposed approach efficiency.

II. EXTRAPOLATION ALGORITHM USING FIR SYNTHESIS FILTER

A. Algorithm

Figure 1 shows a blockdiagram for the proposed extrapolation method. s(n) is an appropriate source signal, and a(n) is the band-limiting filter output signal. The extrapolated signal y(n) is generated through the waveform synthesis digital filter, whose transfer function is H(z). x(n) is the original band-limited signal to be estimated.

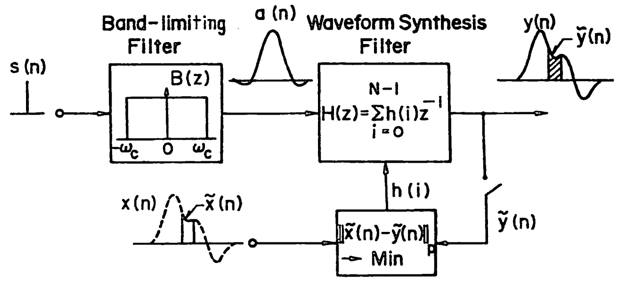

Fig.1 Proposed extrapolation algorithm, using band-limited excited signal and waveform synthesis filter.

Letting X(z) be the z transform of x(n), it satisfies

$$X(e^{j\omega}) = 0, \ \omega \notin \Omega \qquad (1)$$

where Ω means the limited bandwidth. The known part of x(n) is designated by $\tilde{x}$ (n).

$$\tilde{x}(n) = \begin{cases} x(n), & n \in \Lambda \\ 0 & n \notin \Lambda \end{cases} \qquad (2)$$

Λ indicates the time window through which x(n) is observed. Furthermore, $\tilde{y}(n)$ is defined by

$$\tilde{y}(n) = \begin{cases} y(n), & n \in \Lambda \\ 0 \ , & n \notin \Lambda \end{cases} \qquad (3)$$

Letting A(z) be the z transform of a(n), it satisfies

$$A(e^{j\omega}) = 0, \ \ \omega \notin \Omega \qquad (4)$$

By using $a(n)$ as the input signal for the waveform synthesis filter, the band-limited condition is automatically guaranteed. Therefore, the extrapolation problem can be reduced to a filter estimation problem.

In the proposed method, the $H(z)$ coefficients are obtained by solving linear equations, which are formulated so as to minimize the mean square error between $\tilde{x}(n)$ and $\tilde{y}(n)$.

An FIR filter is first considered as the synthesis filter. An IIR filter will be discussed in the next section. $H(z)$ is expressed as,

$$H(z) = \sum_{i=0}^{N-1} h(i)z^{-i} \tag{5}$$

The output signal $y(n)$ is represented as a convolution of $a(n)$ and $h(n)$.

$$y(n) = \sum_{i=0}^{N-1} a(n-i)h(i) \tag{6}$$

Letting the number of $y(n)$ samples be M, the above equation can be represented by a matrix form.

$$\mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ \cdot \\ \cdot \\ y(M-1) \end{bmatrix} = \mathbf{A}\,\mathbf{h} = \begin{bmatrix} a(0), & a(-1), & \cdots, & a(-N+1) \\ a(1), & a(0), & \cdots, & a(-N+2) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a(M-1), & a(M-2), & \cdots, & a(M-N+1) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ \cdot \\ \cdot \\ h(N-1) \end{bmatrix} \tag{7}$$

Furthermore, the output samples in the observation interval Λ are expressed as,

$$\tilde{\mathbf{y}} = \begin{bmatrix} y(k) \\ y(k+1) \\ \cdot \\ \cdot \\ y(k+m-1) \end{bmatrix} = \tilde{\mathbf{A}}\,\mathbf{h} = \begin{bmatrix} a(k), & a(k-1), & \cdots, & a(k-N+1) \\ a(k+1), & a(k), & \cdots, & a(k-N+2) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a(k+m-1), & a(k+m-2), & \cdots, & a(k+m-N+1) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ \cdot \\ \cdot \\ h(N-1) \end{bmatrix} \tag{8}$$

In this case, the elements of Λ are from k to $k+m-1$, and the number of the known samples is m. Although the FIR synthesis filter can extrapolate the fractional data, observed through multiple time windows, mathematical formulations are provided only for a single time window. Numerical examples will be given for both cases.

Letting $\tilde{\mathbf{x}}$ be a vector of $\tilde{x}(n)$, extrapolation error is evaluated by

$$\varepsilon = \| \tilde{\mathbf{x}} - \tilde{\mathbf{y}} \|_2^2 = \| \tilde{\mathbf{x}} - \tilde{\mathbf{A}}\mathbf{h} \|_2^2 \tag{9}$$

where $\| \|_2$ means the L_2 norm. After constructing the ideal band-limited signal matrix $\tilde{\mathbf{A}}$ and the given signal vector $\tilde{\mathbf{x}}$, the least square solution for h can be found by the modified Gram-Schmidt (MGS) procedure [7]. According to the linear equation theory, the maximum order N cannot exceed the number of the known samples.

$$N \leqq m \tag{10}$$

After the synthesis filter coefficients are obtained, the extrapolated waveform $y(n)$ over an entire time interval can be calculated from Eq.(6).

B. Ideal Band-limited Excited Signal

Letting $b(n)$ and $B(z)$ be the impulse response for the band-limitting filter and its z transform, respectively, $a(n)$ and $A(z)$ are expressed as follows:

$$a(n) = \sum_{i=0}^{Ns-1} s(i)b(n-i) \tag{11a}$$

$$A(z) = B(z)S(z) \tag{11b}$$

where, Ns is the number of $s(n)$ samples. Although $s(n)$ is one of parameters used for signal extrapolation, the following impulse $\delta(n)$ is only considered in this paper.

$$\delta(n) = \begin{cases} 1, & n=0 \\ 0 & n \neq 0 \end{cases} \tag{12}$$

Furthermore, the band-limitting filter is assumed to be an ideal lowpass filter. Therefore, the ideal band-limited signal $a(n)$ has zero phase and unity amplitude responses over the given limited band Ω.

$$A(e^{j\omega}) = \begin{cases} 1, & \omega \in \Omega \\ 0, & \omega \notin \Omega \end{cases} \tag{13}$$

$a(n)$ can be considered as an ideal band-limited signal, because it is obtained using $b(n)$ with infinite lengths, and is not truncated in the convolution process given by Eq.(6).

From Eq.(13), $a(n)$ becomes

$$a(n) = 2f_c \frac{\sin(\omega_c nT)}{\omega_c nT} \tag{14}$$

where, ω_c is the edge frequency of the given limited band Ω.

C. Least Square Solution for h(i)

From Eqs.(8) and (9), ε is expressed as,

$$\varepsilon = \sum_{n=0}^{M-1}(\tilde{x}(n) - \tilde{y}(n))^2 = \sum_{n=k}^{k+m-1} (x(n) - y(n))^2 \tag{15a}$$

$$= \sum_{n=k}^{k+m-1} \{x(n) - \sum_{i=0}^{N-1} a(n-i)h(i)\}^2 \quad (15b)$$

Linear equations for h(i), which minimize ε, can be formulated by setting

$$\frac{\partial \varepsilon}{\partial h(i)} = 0, \quad 0 \leq i \leq N-1 \quad (16)$$

From Eqs.(15) and (16), the linear equations become

$$\frac{\partial \varepsilon}{h(i)} = \sum_{n=k}^{k+m-1} 2\{x(n) - \sum_{j=0}^{N-1} a(n-j)h(j)\}a(n-i) = 0, \quad 0 \leq i \leq N-1 \quad (17)$$

The above equation can be written in the following matrix form.

$$\bar{A}^t(\bar{x} - \bar{A}h) = 0 \quad (18)$$

$\bar{A}^t$ means the transposed version of $\bar{A}$. An explicit form for- becomes

$$h = (\bar{A}^t\bar{A})^{-1}\bar{A}^t\bar{A}\,\bar{x} \quad (19)$$

For this least square problem, the orthogonal transformation, i.e., the modified Gram-Scmidt (MGS) procedure, which has been reported to be significantly less sensitive to roundoff errors, can be applied.

D. Hardware Implementation

Equation (18) can be rewritten as,

$$\bar{A}^t\bar{A}h = \bar{A}^t x \quad (20)$$

Since element values in $\bar{A}$ and $\bar{x}$ are already known, $\bar{A}^t\bar{A}$ becomes a constant vector. Furthermore, $\bar{A}^t\bar{A}$ can be broken down into

$$\bar{A}^t\bar{A} = QR \quad (21)$$

where Q satisfies

$$Q^tQ = \mathrm{diag}[d_i], \quad (22)$$

and R is a unit upper triangle matrix.

$$R = \begin{bmatrix} 1 & r_{12} & r_{13} & \cdots & r_{1N} \\ & 1 & r_{23} & \cdots & r_{2N} \\ & & \ddots & & \vdots \\ & 0 & & \ddots & r_{N-1N} \\ & & & & 1 \end{bmatrix} \quad (23)$$

From Eqs.(20) and (21),

$$QRh = \bar{A}^t\bar{x} \quad (24)$$

Multiplying Q^t from the left side, we obtain

$$Rh = \mathrm{diag}[d_i^{-1}]Q^t(\bar{A}^t\bar{x}) \quad (25)$$

If the matrices Q and R could be found, h could be computed by down-up calculation from the above equation.

The MGS procedure is an efficient method for calculating Q and R. Furthermore, the recursive version of the MGS [8] is computationally efficient in real time applications. By using these mathematical tools, the proposed method could be effectively implemented on special or general purpose hardware. A blockdiagram, indicating implementing the proposed method along this direction, is illustrated in Fig.2.

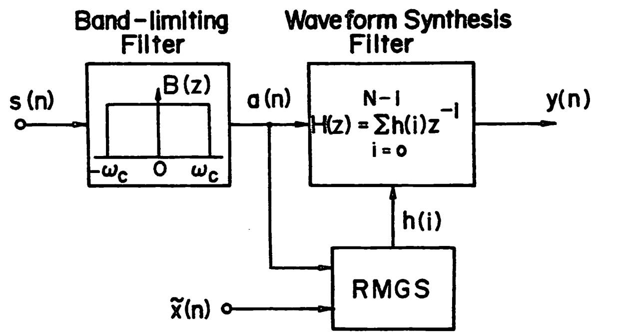

Fig.2 Implementing proposed extrapolation algorithm, based on recursive version of MGS.

III. EXTRAPOLATION ALGORITHM USING IIR SYNTHESIS FILTER

The FIR waveform synthesis filter is extended to an IIR filter in this section. A transfer function is expressed as follows:

$$H(z) = \sum_{i=0}^{N_h-1} h(i)z^{-i} \Big/ \sum_{j=0}^{N_g-1} g(j)z^{-j} \quad (26)$$

The output signal y(n) is obtained through the following difference equations.

$$y(n) = \sum_{i=0}^{N_h-1} h(i)a(n-i) - \sum_{j=1}^{N_g-1} g(j)y(n-j), \quad 0 \leq n \leq M-1 \quad (27)$$

Before solving the above equations for h(i) and g(j), y(n-j) is replaced by x(n-j), as follows:

$$y(n) = \sum_{i=0}^{Nh-1} h(i)a(n-i) - \sum_{j=1}^{Ng-1} g(j)x(n-j), \quad k+N_g \leq n \leq k+m-1 \quad (28)$$

This means that y(n) values, in the region $k \leq n \leq k+Ng-1$, are assumed to be exactly equal to x(n). The extrapolation error in the remaining part, $k+Ng \leq n \leq k+m-1$, becomes

$$\varepsilon = \sum_{n=k+N_g}^{k+m-1} (x(n) - y(n))^2 \quad (29)$$

Substituting y(n) given by Eq.(28) into the above equation, ε is expressed only by h(i) and g(j).

$$\varepsilon = \sum_{n=k+N_g}^{k+m-1} \{x(n) - \sum_{i=0}^{N_h-1} h(i)a(n-i) + \sum_{j=1}^{N_g-1} g(j)x(n-j)\}^2 \quad (30)$$

The linear equations for h(i) and g(j) are formulated by setting

$$\frac{\partial \varepsilon}{\partial h(i)} = 0, \quad 0 \leq i \leq N_{h-1} \quad (31)$$

$$\frac{\partial \varepsilon}{\partial g(j)} = 0, \quad 1 \leq j \leq N_{g-1} \quad (32)$$

The number of the known samples $\tilde{x}(n)$ should be equal to, or greater than, $N_h + N_g-1$. The above formulation is only valid for a single observation time window.

IV. NUMERICAL EXAMPLES

A. FIR Waveform Synthesis Filter

1) Lowpass Signal Extrapolation

The lowpass signal is given by

$$x(n) = \sin(0.075\pi n)-0.075\pi n, \quad 0\leqq n\leqq M-1 \qquad (33)$$

The bandwidth for the excited signal a(n) is chosen to be 0.082π, which is somewhat greater than 0.075π.

$$a(n) = \sin(0.082\pi n)-0.082\pi n, \quad -\infty\leqq n\leqq \infty \qquad (34)$$

Figure 3(a) shows the original signal x(n) and the extrapolated signal y(n) up to 100 samples (M=100). The sampling points for the ob-served data are from 0 to 19 (k=0, m=20). Synthesis filter order N is 18. From Fig.3(a), the extrapolated signal can approximately follow the original signal. The extrapolation error can be further decreased by increasing either the filter order or the observation time interval.

In actual applications, an extrapolation problem regarding the fractional data is often encountered, which are observed in separated time intervals. Figure 3(b) shows simulation results for this example. The known samples $\tilde{x}(n)$ is separated into four time intervals. The total number of the known samples is 20, which is the same as in Fig.3(a). The filter order N is 10. From these results, the proposed approach is more effective for multi-windowed signal extrapolation.

2) Multi-Siusoid Signal Extrapolation

First, let a multi-sinusoid signal be a portion of the sum of three sinusoids,

$$v(n) = \sin(0.03\pi n) + 0.3\sin(0.04\pi n+0.2) + 0.8\sin(0.06\pi n+1.0)\ , \quad 0\leqq n\leqq K-1 \qquad (35)$$

The original signal x(n) is generated by passing v(n) through a lowpass filter, whose bandwidth is 0.075π. The number of v(n) samples is chosen to be 30 (K=30) in this example. The excited signal a(n) is given by Eq.(34). Figure 4(a) shows the original signal x(n) and the extrapolated signal y(n) up to M=100. The filter order N is 18, and the number of the known samples is 20. Figure 4(b) shows the extrapolation results for the fractional data, observed in several time intervals. The parameters are all the same as in Fig.3(a).

B. IIR Synthesis Filter

1) Lowpass Signal Extrapolation

The lowpass signal x(n) and the excited signal a(n) are the same as in Fig.3(a). Figure 5(a) shows the original signal x(n) and the extra-polated signal y(n) up to M=100. The sampling points in the known part are from 0 to 15 (k=0, m=15). The numerator and denominator orders N_h and N_g are 10 and 4, respectively. From this result, the IIR synthesis filter can decrease extrapolation errors, even though using a smaller number of the known samples and a lower-order transfer function than those for the FIR version.

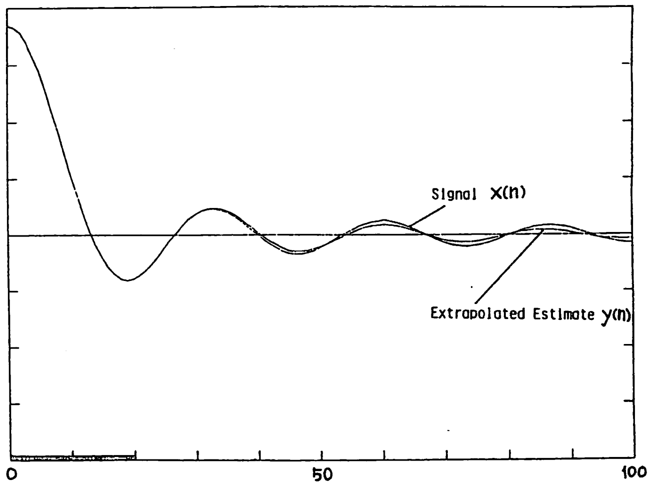

Fig.3 Numerical examples for 18th-order FIR synthesis filter. Original sigal $\tilde{x}(n)$ is lowpass signal. (a) Known x(n) samples are concentrated in the n=0~19 range.

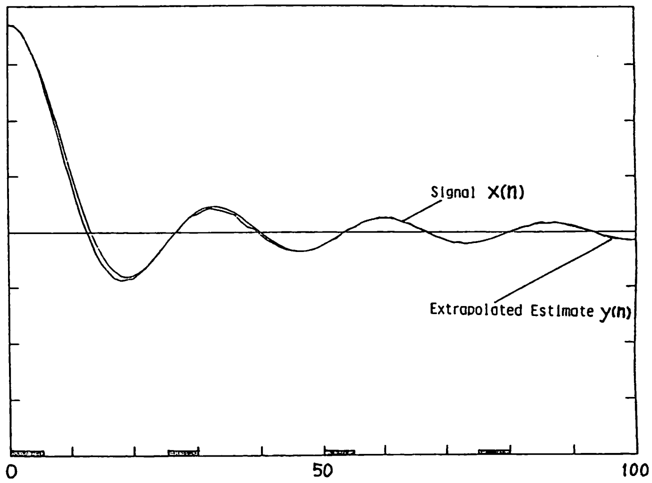

Fig.3(b) Known $\tilde{x}(n)$ samples are distributed in four parts, n=0~4, 25~29, 50~54 and 75~79. The total number of known samples is 20.

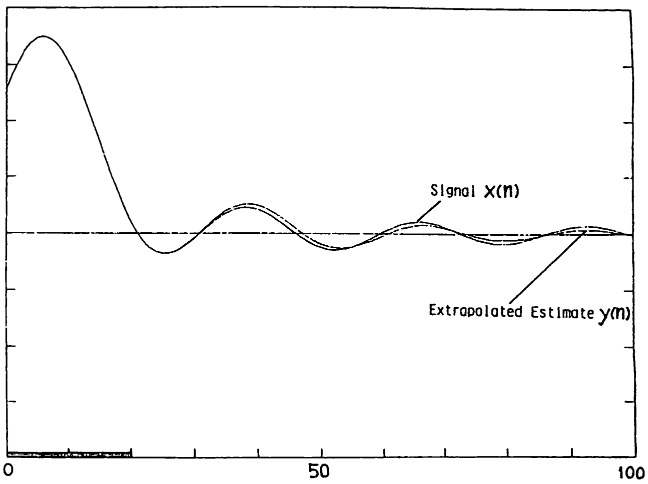

Fig.4 Numerical examples for 18th-order FIR synthesis filter. Original signal $\tilde{x}(n)$ is multi-sinusoid signal. (a) Known $\tilde{x}(n)$ samples are concentrated in the n=0~19 range.

2) Multi-sinusoid Signal Extrapolation

The original signal x(n) and the excited signal a(n) are the same as in Fig.4. The same filter orders are used as in Fig.5(a). The simulation results are shown in Fig.5(b).

In extrapolating both lowpass and multi-sinusoid signals, the IIR synthesis filter is more effective than the FIR version. The FIR synthesis filter has, however, another advantage, i.e., it can be applied to multi-windowed data extrapolation.

V. CONCLUSION

A new signal extraporlation algorithm for discrete band-limited signals has been proposed. The band limited condition is replaced by using the ideal band-limited excited signal. The waveform synthesis digital filter is employed to extrapolate the fractionally observed data. The digital filter coefficients are determined by solving the linear equations, which are formulated so as to minimize the mean square error. Finding the least square solution could be implemented on real time hardware, employing the recursive version of the modified Gram-Schmidt procedure.

From many numerical examples, the proposed algorithm has been recognized to have good performance in extrapolating the fractionally observed data. This algorithm is applicable not only to the extrapolation but also to the spectral estimation.

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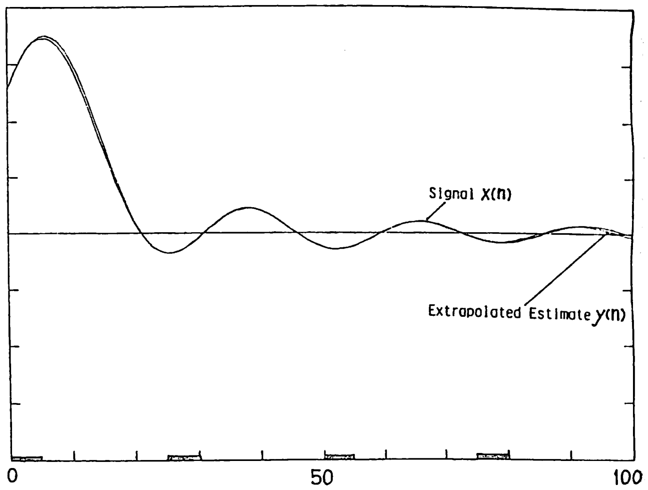

Fig.4(b) Known $\bar{x}(n)$ samples are distributed in four parts. which are the same as in Fig.3(b).

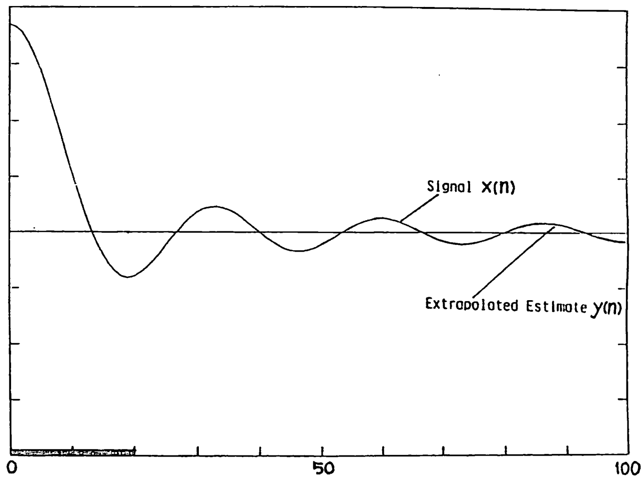

Fig.5 Numerical examples for IIR synthesis filter. Numerator and denominator orders are 10 and 4, respectively. Known $\bar{x}(n)$ samples are concentrated in the n=0~19 range. (a) x(n) is lowpass signal.

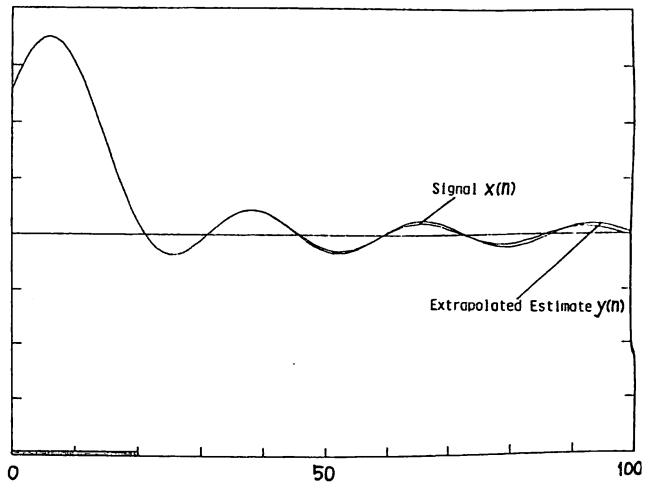

Fig.5(b) x(n) is multi-sinusoid signal.