A Dynamic Traffic Equilibrium Model with Large Discrete Time

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Abstract

Traffic condition in most cities varies significantly within a day, in which static traffic assignment model may not be able to sufficiently represent time-varying congestion phenomena in transportation network analysis. A dynamic traffic assignment (DTA) model needs much computational load and does not necessarily have a unique solution. A continuous time DTA model must be desirable theoretically. However, for practical applications, a discrete time DTA model is also appropriate. In many cases, OD matrix data is not so accurate. The length of discrete time period should be determined according to accuracy of OD data and etc. Too detail description of flow propagation is not necessarily effective. Assume that the length of periods is set from 15 min. to 90min. The length may be greater than that in ordinary discrete DTA models. A dynamic traffic equilibrium model with large discrete time is formulated, and uniqueness of equilibrium is proved.

Keywords: dynamic traffic assignment, network equilibrium, unique solution, large discrete time

1 INTRODUCTION

Traffic condition in most cities varies significantly within a day, in which static traffic assignment model may not be able to sufficiently represent time-varying congestion phenomena in transportation network analysis. On the other hand, a dynamic traffic assignment (DTA) model needs much computational load and does not necessarily have a unique solution in the most models. A continuous time DTA model must be desirable theoretically. However, for practical applications, a discrete time DTA model is also appropriate.

In many cases, OD matrix data is not so accurate. The length of discrete time period should be determined according to accuracy of OD data and etc. Too detail description of flow propagation is not necessarily effective. Therefore, for practical applications, the period length may be from 15 min to 90 min. in many cases. Needless to say, we can set the length much shorter if OD demand data are accurate and dynamically detail. The object of this study is the case that OD data are not detail, and the period (or the length of discrete time) is not so short (approximate from 15 min to 90min.).

In this case, one of the methods is to formulate a static equilibrium in each period separately. This method does not consider flow propagation or flow dynamics at all. When heavy congestion occurs, the congestion remains in the next period at least partially. The separate static equilibrium approach is too rough to describe the network flow dynamics. In this study, we adopt a large discrete time approach. The large discrete time approach is basically to formulate a static equilibrium in each period (large discrete time), but considers flow propagation between periods. A large discrete time is called “period” in this paper. The flow propagation that is considered is that overflow in a period is propagated to the next period. An ordinary discrete time dynamic model and the model with large discrete time are similar. The model with large discrete time has relatively long time period, say 15 min. through 90min. In the model, a static network equilibrium is reached in each period and flow propagations is considered between periods.

In this paper, a new concept of traffic equilibrium for large discrete time is defined, and a model is formulated based on the concept. Then, existence and uniqueness of the model are examined.

2 FORMULATION

2.1 Flow conservation

The traffic flow on a link will be carried over to the next time period representing the propagation of the congestion from one period to another. The flow conservation condition is formulated as:

$$\sum_{j \in N_i^{out}} x_{ijnt} = q_{int} + \sum_{k \in N_i^{in}} z_{kint} \quad \forall i \in N_{-n}, n \in D, t \in T \tag{1}$$

$$q_{int} = d_{int} + \sum_{k \in N_i^{in}} (x_{kint-1} - z_{kint-1}) \tag{2}$$

where x_{ijnt} denotes the inflow to Link ij in Period t bound for Node n, z_{ijnt} the outflow from Link ij, d_{int} the travel demand bound for Node n which departs in Period t, N_{-n} the set of all nodes except Node n, D the set of destination nodes, T the set of time periods, N_i^{out} the set of end nodes of the links that are connected from Node i, and N_i^{in} the set of start nodes of the links that connect to Node i. Let A denote the set of links. ij present the link between Node i and Node j and $(i, j) \in A_{-n}$ means the set of nodes or links except the links whose end node is Node n or Node n.

Assume simply that link travel time is a function of its inflow. The flow which does not reach the end of the link is carried over to the next period on that link. Furthermore, assume that the flow that does not reach the destination re-starts from the end of the link on which the flow is carried over in the next period.

2.2 Flow dynamics between periods

As mentioned in the introduction section, in this study, traffic equilibrium is reached in each period, and the dynamics between periods is considered. The dynamics which is modeled is that the flow which cannot exit from the link within the period remains on that link. Thus, the flow which cannot reach the destination is propagated to the next period. We shall call this a "remaining flow." The dynamics in this study is that the remaining flows are propagated to the next period.

The difference between the inflow and outflow is the remaining flow, that is, a flow propagated to the next period. Let the remaining flow to the next period denote y_{ijnt}. Then, $y_{ijnt} = x_{ijnt} - z_{ijnt}$. Assume that the remaining flow exit the link in the (next) period, that is, the period after the flow enter the link. Namely, the inflow exits the link in the period and the next period. The remaining flow, y_{ijnt}, has already passed the start node of Link ij, Node i, at the end in Period t, where Link ij is the link between Node i and Node j.

In this study, the period length is not so short as an ordinary DTA model's. Assume simply that link travel time is a function of its inflow. This is not necessarily appropriate from the standpoint of traffic flow theory, but we do not need too accurate travel time, comparing the model framework and OD matrix data. In this study, assume $y_{ijt} = g_{ij}(x_{ijt})$, that is, the remaining flow is a function of the inflow only. Also, assume that travel time function are strictly increasing and convex, and $g_{ij}(x_{ijt})$ is also increasing.

2.3 A new traffic equilibrium

In continuous time DTA studies, the travel time the flow experiences is equilibrated. In some of them, equilibrium is modeled as $c_{ij}(t) + \tau_{jn}(t + c_{ij}(t)) - \tau_{in}(t) = 0$ if $x_{ijn}(t) > 0$ and $c_{ij}(t) + \tau_{jn}(t) - \tau_{in}(t) \geq 0$ if $x_{ijn}(t) = 0$ $(\forall i \in N_{-n})$ where $c_{ij}(t)$ denote travel time of Link ij that the flow which departs from Node i at Time t exeriences (will experience), $x_{ijn}(t)$ the inflow to Link ij which deaprts at Time t bound for Node n, $\tau_{in}(t)$ the minimum travel cost from Node i to Node n

(destination) in Time t. This means that Link ij is on the route which has the minimum travel time.

In this study, a part of the inflow cannot get out of the link and must take the travel cost from the end of the link to the destination in the next period while the other part of the flow takes it in this period. Thus, the minimum travel cost should be defined taking the travel cost in the next period into consideration. The new minimum travel cost in this study is:

$$\mu_{ijnt} \equiv \begin{cases} \dfrac{z_{ijnt}}{x_{ijnt}}\tau_{jnt} + \left(1 - \dfrac{z_{ijnt}}{x_{ijnt}}\right)\tau_{jnt+1} & \textit{if } x_{ijnt} > 0 \\ \tau_{jnt} & \textit{if } x_{ijnt} = 0 \end{cases}. \tag{3}$$

where τ_{int} denotes the (mean) minimum travel time between Node i and Node n. This is a weighted average of minimum travel costs in Period t and following periods because a part of the flow travels in Period 2 or the following periods.

The equilibrium can be formulated as:

$$\begin{aligned} c_{ijt} + \mu_{ijnt} - \tau_{int} = 0 \quad \textit{if } x_{ijnt} > 0 \\ c_{ijt} + \mu_{ijnt} - \tau_{int} \geq 0 \quad \textit{if } x_{ijnt} = 0 \end{aligned}. \tag{4}$$

This means that a link is on the route which has the minimum (mean) travel time.

The dynamic equilibrium model is formulated as the complementarity problem which satisfies:

$$x_{ijnt}\left(c_{ijt} + \mu_{ijnt} - \tau_{int}\right) = 0 \qquad \forall(i,j) \in A_{-n},\ n \in D,\ t \in T \tag{5}$$

$$x_{ijnt} \geq 0,\ c_{ijt} + \mu_{ijnt} - \tau_{int} \geq 0 \qquad \forall(i,j) \in A_{-n},\ n \in D,\ t \in T \tag{6}$$

$$\tau_{int}\left(u_{int} - q_{int} - v_{int}\right) = 0,\ \tau_{int} \geq 0 \qquad \forall(i,j) \in A_{-n},\ n \in D,\ t \in T \tag{7}$$

$$u_{int} - q_{int} - v_{int} \geq 0 \qquad \forall(i,j) \in A_{-n},\ n \in D,\ t \in T \tag{8}$$

where

$$u_{int} = \sum_{j \in N_i^{out}} x_{ijnt} \tag{9}$$

$$v_{int} = \sum_{k \in N_i^{in}} z_{kint} \tag{10}$$

$$q_{int} = d_{int} + \sum_{k \in N_i^{in}} (x_{kint-1} - z_{kint-1}) \tag{11}$$

Fig. 1 shows an example of the minimum travel time on the route which consists of 3 (series connected) links. The mean minimum travel time can be calculated using Eq. (6).

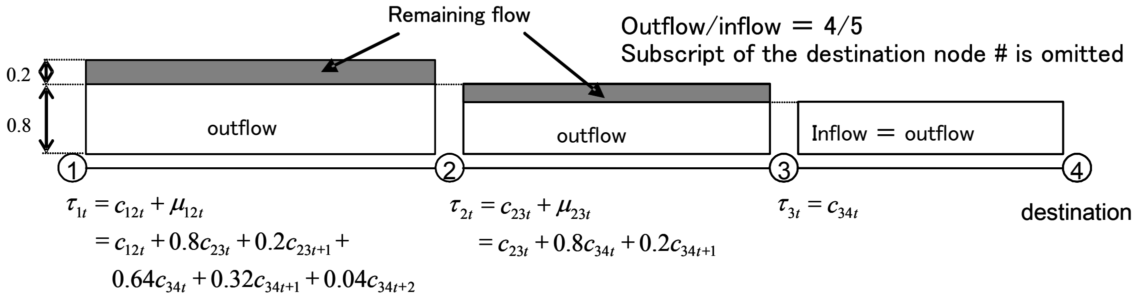


Fig. 1 An example of the minimum travel time

3 EXISTENCE AND UNIQUENESS OF EQUILIBRIUM

The proof of existence and uniqueness of equilibrium in this study is similar to Aashtiani & Magnanti (1981) and Wie et al. (2002). Existence of equilibrium can be proved using the same manner of their paper. In this section, uniqueness of equilibrium will be proved.

At first, we assume that the model has two different solutions, $(\mathbf{x}^*, \boldsymbol{\tau}^*)^T$ and $(\mathbf{x}^\circ, \boldsymbol{\tau}^\circ)^T$. Then, the following equations hold:

$$x^*_{ijnt}\left(c^*_{ijt} + \mu^*_{ijnt} - \tau^*_{int}\right) = 0 \tag{12}$$
$$x^\circ_{ijnt}\left(c^\circ_{ijt} + \mu^\circ_{ijnt} - \tau^\circ_{int}\right) = 0 \tag{13}$$

Summing the above two equations yields:

$$\left(x^*_{ijnt} - x^\circ_{ijnt}\right)\left(c^*_{ijt} - c^\circ_{ijt} + \mu^*_{ijnt} - \mu^\circ_{ijnt} - \tau^*_{int} + \tau^\circ_{int}\right) + x^\circ_{ijnt}\left(c^*_{ijt} + \mu^*_{ijnt} - \tau^*_{int}\right) + x^*_{ijnt}\left(c^\circ_{ijt} + \mu^\circ_{ijnt} - \tau^\circ_{int}\right) = 0 \tag{14}$$

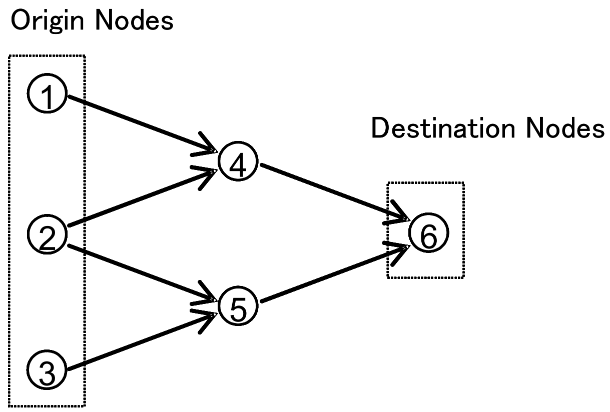
Let $\vec{x}_{ijnt} = x^*_{ijnt} - x^\circ_{ijnt}$, $\vec{c}_{ijt} = c^*_{ijt} - c^\circ_{ijt}$, $\vec{\mu}_{ijnt} = \mu^*_{ijnt} - \mu^\circ_{ijnt}$, $\vec{\tau}_{int} = \tau^*_{int} - \tau^\circ_{int}$, and $\vec{q}_{ijnt} = q^*_{ijnt} - q^\circ_{ijnt}$. By Eq. (5), $x^*_{ijnt} \geq 0$, $x^\circ_{ijnt} \geq 0$, $c^*_{ijt} + \mu^*_{ijnt} - \tau^*_{int} \geq 0$, $c^\circ_{ijt} + \mu^\circ_{ijnt} - \tau^\circ_{int} \geq 0$, the second and third terms of the above equation is not negative. Therefore, using $\vec{x}_{ijnt}$, $\vec{c}_{ijt}$, $\vec{\mu}_{ijnt}$ and $\vec{\tau}_{int}$, the equation can be organized as:

$$\vec{x}_{ijnt}\left(\vec{c}_{ijt} + \vec{\mu}_{ijnt} - \vec{\tau}_{int}\right) \leq 0 \tag{15}$$

Similarly,

$$\vec{\tau}_{int}\left(\vec{u}_{int} - \vec{q}_{int} - \vec{v}_{int}\right) \leq 0 \tag{16}$$

can also be derived. We sum Eq. (15) with respect to $j \in N_i^{out}$, and add it to Eq. (16). Then, we obtain:



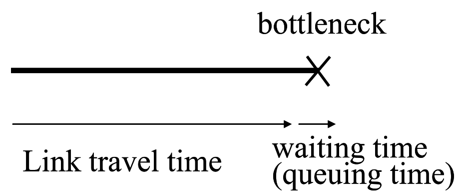


Fig. 2 The example network

Fig. 3 Link ravel time in the example

$$\sum_{j\in N_i^{out}} \vec{x}_{ijnt}\left(\vec{c}_{ijt}+\vec{\mu}_{ijnt}\right) - \vec{\tau}_{int}\,\vec{q}_{int} - \vec{\tau}_{int}\sum_{k\in N_i^{in}} \vec{z}_{kint} \le 0 \tag{17}$$

Note that we used Eq (9) and (10). Substitute Eq. (3) for Eq. (17) yields:

$$\sum_{ij\in A} \vec{x}_{ijnt}\,\vec{c}_{ijt} + \sum_{ij\in A} \vec{\tau}_{jnt+1}\left(\vec{x}_{ijnt}-\vec{z}_{ijnt}\right) - \sum_{i\in N} \vec{\tau}_{int}\,\vec{q}_{int} \le 0 \tag{18}$$

where $\vec{x}_{nknt}, \vec{z}_{nknt} = 0$ ($k \in N_n^{out}$) which means that no flow exits from the destination, because $\sum_i \sum_j x_{ijnt}\ \mu_{ijnt} = \sum_i \sum_j \tau_{jnt}\ z_{ijnt} + \sum_i \sum_j \tau_{jnt+1}\ (x_{ijnt} - z_{ijnt})$, and $\sum_i \sum_j \tau_{jnt}\ z_{ijnt} = \sum_i\ \tau_{int}\ \sum_k\ z_{kint}$. Summing Eq. (18) with respect to $\forall n \in D$, $t \in T$ gives:

$$\sum_{ij\in A}\sum_{t\in T} \vec{c}_{ijt}\vec{x}_{ijt} + \sum_{ij\in A}\sum_{n\in D}\sum_{t\in T} \vec{\tau}_{jnt+1}\left(\vec{x}_{ijnt}-\vec{z}_{ijnt}\right) - \sum_{i\in N}\sum_{n\in D}\sum_{t\in T} \vec{\tau}_{int}\,\vec{q}_{int} \le 0 \tag{19}$$

where $\vec{x}_{ijt} = x^*_{ijt} - x^\circ_{ijt}$, $x^*_{ijt} = \sum_{n\in D} x^*_{ijnt}$, $x^\circ_{ijt} = \sum_{n\in D} x^\circ_{ijnt}$.

OD demands are constant, and $\vec{d}_{int} (= d^*_{int} - d^\circ_{int}) = 0$. By Eq. (11), $\vec{q}_{int} = \sum_k (\vec{x}_{kint-1} - \vec{z}_{kint-1})$. Substituting this for Eq. (19) gives:

$$\sum_{ij\in A}\sum_{t\in T}\left(c^*_{ijt}-c^\circ_{ijt}\right)\left(x^*_{ijt}-x^\circ_{ijt}\right) \le 0 \tag{20}$$

This is contradicts the convexity of travel time functions. Thus, link inflows in each period are unique. Note that $\{x_{ijt} \mid \forall(i,j)\in A, t\in T\}$ is unique, and x_{ijnt} is not necessarily unique.

4 EXAMPLE

We apply the model to a simple network. The network has 6 nodes and 6 links as shown in Fig. 2. Each link consists of a travel part and a bottleneck part. The travel time of the travel

Table 1 Link parmances

	free-flow time	capacity
Link 14	10	150
Link 24	10	175
Link 25	10	125
Link 35	10	150
Link 46	10	200
Link 56	10	200

Table 2 OD demands

	Period 1	Period 2
1→6	70	60
2→6	350	300
3→6	70	60

Table 3. Inflows, travel times and remaning flows in the example

	Period 1				Period 2			
	Link 24	Link 25	Link 46	Link 56	Link 24	Link 25	Link 46	Link 56
inflow	189.8	160.2	245.0	195.0	163.0	137.0	237.8	220.2
travel time	16.46	31.98	26.88	11.36	11.19	16.77	24.46	18.87
remaning flow	14.8	35.2	45.0	0.0	0.0	12.0	37.8	20.2

part and bottleneck are given by the BPR-type function and vertical queue. In this example, the capacity of travel part and bottleneck part are equal for simplicity. Table 1 shows link's performances. OD pairs are Node 1 and 6, Node 2 and 6, and Node 3 and 6. The flow from Node 2 has route choice, but the others do not. Demands are written in Table 2.

Table 3 shows the results of the example assignment. The equilibrium is formulated by Eq. (7). A part of the flow which departs at Node chooses Link 24. This inflow to Link 24 does not necessarily exit from Link 24 within Period 1. 14.8 of the inflow remain on Link 24 and travels on Link 46 in Period 2 while 175.0 exits from Link 1 in Period 1. The mean travel time which takes Link 24 in Period 1 is: 16.46 + (175.0/189.8)*26.88 + (14.8/189.8)*24.46 = 43.84. Similarly, the mean travel time of the flow which takes Link 25 is 31.98 + (125.0/160.2)*11.36 + (35.2/160.2)*18.87 = 43.84. Thus, the equilibrium is reached.

5 CONCLUSIONS

Traffic condition in most cities varies significantly within a day, in which static traffic assignment model may not be able to sufficiently represent time-varying congestion phenomena in transportation network analysis. On the other hand, a dynamic traffic assignment (DTA) model needs much computational load and does not necessarily have a unique solution in the most models. A continuous time DTA model must be desirable theoretically, but, for practical applications, a discrete time DTA model is also appropriate.

In this study, the case that the period (or the length of discrete time) is not so short (approximate from 15 min to 90min.) is considered. The new concept of traffic equilibrium is defined for these cases, and a dynamic traffic equilibrium model with large discrete time is formulated based on the concept. Then, existence and uniqueness of the model are examined. The model is also applied to a simple network, and its characteristics are examined.

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