LAGUERRE AND DISK POLYNOMIAL EXPANSIONS WITH NONNEGATIVE COEFFICIENTS

YUICHI KANJIN

Abstract. We establish Wiener type theorems and Paley type theorems for Laguerre polynomial expansions and disk polynomial expansions with nonnegative coefficients.

1. Introduction

A well-known theorem on functions with positive Fourier coefficients given by Norbert Wiener (see [4, pp.242-250] and [19, §§1-2]) is the following:

[A] Wiener's theorem . *Let $f \in L^1(-\pi,\pi)$ be a function satisfying $\hat{f}(n) \geq 0$ for every $n \in \mathbb{Z}$, where $\hat{f}(n) = (1/(2\pi)) \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta$. If there exists a constant $\delta > 0$ such that $\int_{-\delta}^{\delta} |f(\theta)|^2\, d\theta < \infty$, then $\int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta < \infty$.*

On functions with positive Fourier coefficients satisfying $\operatorname{ess\,sup}_{|\theta|<\delta}|f(\theta)| < \infty$ with some $\delta > 0$, we have the following which is a part of the results of Paley [18]:

[B] Paley's theorem . *Let $f \in L^1(-\pi,\pi)$ be an even function satisfying $\hat{f}(n) \geq 0$ for every n. If $\operatorname{ess\,sup}_{|\theta|<\delta}|f(\theta)| < \infty$ with some $\delta > 0$, then $\sum_{n=-\infty}^{\infty} \hat{f}(n) < \infty$.*

Recently, Mhaskar and Tikhonov [17] extended these two theorems to the Jacobi polynomial expansions. Let us state an essential part of their results. Let $R_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomials of order $\alpha, \beta > -1$ with the normalization $R_n^{(\alpha,\beta)}(1) = 1$, that is, the orthogonal polynomials $p_n(x)$ on the interval $[-1,1]$ with respect to the weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ satisfying $p_n(1) = 1$. It is known that $R_n^{(-1/2,-1/2)}(\cos\theta) = \cos n\theta$. A function f on $[-1,1]$ is formally expanded: $f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)R_n^{(\alpha,\beta)}(x)$. Here, $\hat{f}(n)$ is the Fourier-Jacobi coefficient of f defined by

$$\hat{f}(n) = \rho_n^{-1} \int_{-1}^{1} f(x)R_n^{(\alpha,\beta)}(x)w_{\alpha,\beta}(x)\, dx, \quad \rho_n = \int_{-1}^{1} |R_n^{(\alpha,\beta)}(x)|^2 w_{\alpha,\beta}(x)\, dx.$$

[C] ([17])**.** *Let $f \in L^1([-1,1], w_{\alpha,\beta})$. Suppose that every Fourier-Jacobi coefficient $\hat{f}(n)$ is nonnegative. Then the following* (i) *and* (ii) *hold.*

(i) *If there exists a constant $\delta > 0$ such that $\int_{1-\delta}^{1} |f(x)|^2 w_{\alpha,\beta}(x)\, dx < \infty$, then $f \in L^2([-1,1], w_{\alpha,\beta}(x))$.*

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(ii) *If there exists a constant* $\delta > 0$ *such that* $\operatorname{ess\,sup}_{1-\delta<x<1}|f(x)| < \infty$*, then* $\sum_{n=0}^{\infty} \hat{f}(n) < \infty$.

Actually, Mhaskar and Tikhonov have obtained a more general Wiener type theorem ([17, Theorem 3.1]) by using the notion of solid space. A subspace $X \subset L^1([-1,1], w_{\alpha,\beta})$ is called solid if $f \in L^1([-1,1], w_{\alpha,\beta})$, $g \in X$ and $|\hat{f}(n)| \le \hat{g}(n)$ for every n imply $f \in X$. Their results [**C**] suggest that it is interesting to consider Wiener type and Paley type theorems in other orthogonal polynomial expansions.

In this paper, we shall establish these types of theorems in the Laguerre polynomial expansions (Theorem 1 in §2.2) and the disk polynomial expansions (Theorem 2 in §3.2). The disk polynomials are orthogonal polynomials with two variables (cf. [5, 2.4.3 and p.62]). The Laguerre polynomials are orthogonal on the non-compact interval $[0,\infty)$. Kawazoe, Onoe and Tachizawa [14, §2] constructed a function $f \in L^1(\mathbb{R})$ with nonnegative Fourier transform $\hat{f}(\xi) \ge 0$ such that $\int_{-\delta}^{\delta} |f(x)|^2\, dx < \infty$ with some $\delta > 0$ and $f \notin L^2(\mathbb{R})$, which is in contrast to our Wiener type theorem for the Laguerre case.

Related results and further references are found in [1], [2], [7], [8], [13], [16] and [21].

We shall deal with Laguerre polynomial expansions with nonnegative Fourier-Laguerre coefficients in §2. In §2.1, we shall state known results on the Laguerre polynomials and prepare two lemmas which are essential in our proofs of Wiener type and Paley type theorems. Those theorems and other results will be proved in §2.2. In §3, we shall discuss the disk polynomial case in the same order as the Laguerre case. We set an addendum at the end of the paper for proofs of some results on the disk polynomials.

2. Laguerre polynomial expansions

A Wiener type theorem and a Paley type theorem for the Laguerre polynomial expansions will be given in this section. *We suppose throughout this section that the parameter* α *satisfies* $\alpha \ge 0$ and the functions we treat are real-valued. We shall work on the following spaces:

$$L^p_\alpha = \begin{cases} \left\{ f\,;\, \|f\|_p = \left(\int_0^\infty |f(x)e^{-x/2}|^p x^\alpha\, dx\right)^{1/p} < \infty \right\}, & 1 \le p < \infty, \\ \left\{ f\,;\, \|f\|_\infty = \text{ess sup}_{x>0}\, |f(x)e^{-x/2}| < \infty \right\}, & p = \infty. \end{cases}$$

As the above, the weighted norms are denoted by $\|f\|_p$ without the subscript α.

2.1. Preparations.

In this subsection, we summarize some facts and results without proofs which are referred mainly to [9], and we shall give two lemmas which will be used for proving our theorems.

Let $L_n^{(\alpha)}(x)$ be the Laguerre polynomial of degree $n = 0, 1, 2, \ldots$, which is given by the following Rodrigues' formula

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \left(\frac{d}{dx}\right)^n (e^{-x} x^{n+\alpha}).$$

The orthogonality is

$$\int_0^\infty L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\, dx = \Gamma(\alpha+1)\binom{n+\alpha}{n}\delta_{mn}, \qquad m, n = 0, 1, 2, \ldots$$

and the values at $x = 0$ are

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}.$$

We denote the normalized Laguerre polynomials by

$$R_n^{(\alpha)}(x) = L_n^{(\alpha)}(x)/L_n^{(\alpha)}(0),$$

and then the system $\{R_n^{(\alpha)}\}_{n=0}^{\infty}$ is complete and orthogonal in L^2_α. The polynomials satisfy the following inequality ([6, 10.18(14)]):

$$|R_n^{(\alpha)}(x)e^{-x/2}| \leq 1. \tag{1}$$

We define the Fourier Laguerre coefficients $\{\hat{f}(n)\}_{n=0}^{\infty}$ by

$$\hat{f}(n) = \int_0^\infty f(x)R_n^{(\alpha)}(x)e^{-x}x^\alpha\, dx,$$

which satisfy

$$|\hat{f}(n)| \leq \|f\|_1; \qquad |\hat{f}(n)| \leq \|f\|_2\|R_n^{(\alpha)}\|_2.$$

A function $f(x)$ on the interval $[0, \infty)$ is formally expanded as follows:

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n)h_n^{(\alpha)}R_n^{(\alpha)}(x) = \frac{1}{\Gamma(\alpha+1)}\sum_{n=0}^{\infty}\hat{f}(n)L_n^{(\alpha)}(x),$$

where

$$h_n^{(\alpha)} = \frac{1}{\|R_n^{(\alpha)}(x)\|_2^2} = \frac{1}{\Gamma(\alpha+1)}\binom{n+\alpha}{n} \sim n^\alpha.$$

The linearization coefficients

$$\gamma(k, m, n; \alpha) = \int_0^\infty R_k^{(\alpha)}(x)R_m^{(\alpha)}(x)R_n^{(\alpha)}(x)e^{-2x}x^\alpha\, dx$$

satisfy the following [3, Theorem 1, (4.2) and (4.4)]:

$$e^{-x}R_n^{(\alpha)}(x)e^{-x}R_m^{(\alpha)}(x) = \sum_{k=0}^{\infty}\gamma(k, m, n; \alpha)h_k^{(\alpha)}e^{-x}R_k^{(\alpha)}(x), \quad x \geq 0, \tag{2}$$

$$\gamma(k, m, n; \alpha) \geq 0, \qquad \sum_{k=0}^{\infty} h_k^{(\alpha)}\gamma(k, m, n; \alpha) = 1.$$

Let $1 \leq p \leq \infty$. For $f \in L^p_\alpha$, $g \in L^1_\alpha$, the convolution $f * g$ is defined by

$$f * g(t) = \int_0^\infty T_t^\alpha(f; x)g(x)e^{-x}x^\alpha\, dx, \quad t \geq 0,$$

where T_t^α denotes the Laguerre translation operator given by

$$T_t^\alpha(f; x) = \frac{\Gamma(\alpha+1)2^\alpha}{\sqrt{2\pi}}\int_0^\pi f(x + t + 2\sqrt{xt}\cos\theta)e^{-\sqrt{xt}\cos\theta}\frac{J_{\alpha-1/2}(\sqrt{xt}\sin\theta)}{(\sqrt{xt}\sin\theta)^{\alpha-1/2}}\sin^{2\alpha}\theta\, d\theta$$

for $x, t > 0$, $T_t^\alpha(f; 0) = f(t)$ for $t > 0$, $T_0^\alpha(f; x) = f(x)$ for $x \geq 0$. Then the following inequalities hold:

$$\|T_t^\alpha f\|_p \leq e^{t/2}\|f\|_p \ (t \geq 0); \qquad \|f * g\|_p \leq \|f\|_p\|g\|_1.$$

Further the operator T_t^α satisfies

$$T_t^\alpha(R_n^{(\alpha)}; x) = R_n^{(\alpha)}(x)R_n^{(\alpha)}(t), \quad x, t \geq 0.$$

For $f \in L^p_\alpha,\ g \in L^q_\alpha,\ 1 \le p, q \le \infty$ with $1/p + 1/q \ge 1$, the convolutions $f * g$ and $g * f$ exist and

$$\widehat{T^\alpha_t f}(n) = \hat{f}(n) R^{(\alpha)}_n(t) \quad (t \ge 0); \qquad \widehat{f * g}(n) = \hat{f}(n)\hat{g}(n).$$

The Poisson integrals of a function $f \in L^p_\alpha, 1 \le p \le \infty$ is defined by

$$P^{(\alpha)}_r(f; x) = (f * P^{(\alpha)}_r)(x) = \sum_{n=0}^{\infty} r^n \hat{f}(n) h^{(\alpha)}_n R^{(\alpha)}_n(x),\ 0 \le r < 1,\ x \ge 0,$$

where

$$P^{(\alpha)}_r(x) = \sum_{n=0}^{\infty} r^n h^{(\alpha)}_n R^{(\alpha)}_n(x) = \frac{e^{xr/(r-1)}}{\Gamma(\alpha+1)(1-r)^{\alpha+1}},$$

which satisfy

$$\|P^{(\alpha)}_r\|_1 \le \left(\frac{2}{1+r}\right)^{\alpha+1}, \quad 0 \le r < 1. \tag{3}$$

Parseval's formula is as follows:

$$\int_0^\infty f(x)g(x)e^{-x}x^\alpha\, dx = \sum_{n=0}^{\infty} h^{(\alpha)}_n \hat{f}(n)\hat{g}(n)$$

for functions $f, g \in L^2_\alpha$.

We now come to the lemmas which play an essential role to prove our Wiener type and Paley type theorems.

Lemma 1. *Let $\delta > 0$. Then there exists a function ϕ_δ on $[0, \infty)$ such that* $\operatorname{supp} \phi_\delta \subset [0, \delta)$, $\widehat{\phi_\delta}(0) = 1$, $\widehat{\phi_\delta}(n) \ge 0$ *for every n, and $\widehat{\phi_\delta}(n) = O(n^{-k})$, $n \to \infty$ for any positive integer k.*

Proof. We choose a function $g_\delta \in C^\infty(0, \infty)$ such that $g_\delta \ge 0$, $\operatorname{supp} g_\delta \subset (0, \delta/4)$, and

$$\widehat{g_\delta}(0) = \int_0^\infty g_\delta(x)e^{-x}x^\alpha\, dx = 1.$$

Then we put

$$\phi_\delta(t) = g_\delta * g_\delta(t) = \int_0^{\delta/4} T^\alpha_t(g_\delta; x) g_\delta(x) e^{-x}x^\alpha\, dx.$$

We show first that $\operatorname{supp} \phi_\delta \subset [0, \delta)$. We see that

$$\begin{aligned} x + t + \sqrt{xt}\cos\theta &\ge x + t - \sqrt{xt} \ge (\sqrt{t} - \sqrt{x})^2, \\ &\ge (\sqrt{\delta} - \sqrt{\delta/4})^2 = \delta/4 \end{aligned}$$

for $t \ge \delta$, $x \le \delta/4$ and $0 \le \theta \le \pi$. It follows from the definition of the Laguerre translation operator and $\operatorname{supp} g_\delta \subset (0, \delta/4)$ that $T^\alpha_t(g_\delta; x) = 0$ for $t \ge \delta$ and $x \le \delta/4$, which implies $\operatorname{supp} \phi_\delta \subset [0, \delta)$. By $\widehat{\phi_\delta}(n) = \widehat{g_\delta}^2(n)$, we have $\widehat{\phi_\delta}(0) = 1$ and $\widehat{\phi_\delta}(n) \ge 0$ for every n. Since $g_\delta \in C^\infty(0, \infty)$ and $\operatorname{supp} g_\delta \subset (0, \delta/4)$, it follows from integration by parts that $\widehat{g_\delta}(n) = O(n^{-k})$, $n \to \infty$ for any positive integer k (cf., e.g., [12, Lemma 1]), so do the coefficients $\widehat{\phi_\delta}(n)$. □

Remark 1. (i) The function ϕ_δ of this type was used by Mhaskar and Tikhonov [17] and played an important role in their proofs of Wiener and Paley type theorems for the Jacobi expansions.

(ii) From the definitions of the Laguerre translation operator and the convolution, it is easy to see that the function ϕ_δ is continuous on $[0,\infty)$. It follows from (1) that the series $\sum_{n=0}^{\infty} h_n^{(\alpha)} \hat{\phi}_\delta(n) R_n^{(\alpha)}(x) e^{-x/2}$ converges uniformly to a continuous function $g(x)$ on $[0,\infty)$, and for every $x \in [0,\infty)$ the series $\sum_{n=0}^{\infty} h_n^{(\alpha)} \hat{\phi}_\delta(n) R_n^{(\alpha)}(x)$ converges to $g(x)e^{x/2}$. On the other hand, the Poisson integral $P_r^{(\alpha)}(\phi_\delta; \cdot)$ of ϕ_δ converges to ϕ_δ in $L_\alpha^p, 1 \le p < \infty$. Standard arguments lead us to $g(x)e^{x/2} = \phi_\delta(x)$ for every $x \in [0,\infty)$. Therefore we have that

$$\phi_\delta(x) e^{-x/2} = \sum_{n=0}^{\infty} h_n^{(\alpha)} \hat{\phi}_\delta(n) R_n^{(\alpha)}(x) e^{-x/2},$$

where the series converges uniformly on $[0,\infty)$, and

$$\phi_\delta(x) = \sum_{n=0}^{\infty} h_n^{(\alpha)} \hat{\phi}_\delta(n) R_n^{(\alpha)}(x),$$

where the series converges for every $x \in [0,\infty)$. Further, we can see that $\phi_\delta \in C^\infty(0,\infty)$ by using the formula $(d/dx)L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x)$.

Lemma 2. *For $\delta > 0$, let ϕ_δ be the function in Lemma 1. Suppose that $f \in L_\alpha^1$ and $\hat{f}(n) \ge 0$ for every n. Then,*

$$0 \le \widehat{f\check{e}}(n) \le \Gamma(\alpha+1)(f\check{e}\phi_\delta)\widehat{\ }(n), \tag{4}$$

for every n, where $\check{e}(x) = e^{-x}$.

Proof. Since

$$\int_0^\infty |f(x)e^{-x} R_m^{(\alpha)}(x) R_n^{(\alpha)}(x)| e^{-x} x^\alpha \, dx \le \int_0^\infty |f(x)| e^{-x} x^\alpha \, dx < \infty$$

by (1), it follows that

$$\begin{aligned} (f\check{e}\phi_\delta)\widehat{\ }(n) &= \int_0^\infty f(x)e^{-x} \phi_\delta(x) R_n^{(\alpha)}(x) e^{-x} x^\alpha \, dx \\ &= \sum_{m=0}^{\infty} h_m^{(\alpha)} \widehat{\phi_\delta}(m) \int_0^\infty f(x)e^{-x} R_m^{(\alpha)}(x) R_n^{(\alpha)}(x) e^{-x} x^\alpha \, dx. \end{aligned} \tag{5}$$

We put

$$I_0(m,n) = \int_0^\infty f(x)e^{-x} R_m^{(\alpha)}(x) R_n^{(\alpha)}(x) e^{-x} x^\alpha \, dx.$$

Then we have by (2) that

$$I_0(m,n) = \sum_{k=0}^{\infty} h_k^{(\alpha)} \gamma(k,m,n;\alpha) \hat{f}(k), \tag{6}$$

which is justified by

$$\int_0^\infty |f(x) R_k^{(\alpha)}(x)| e^{-x} x^\alpha \, dx \le \int_0^\infty |f(x)| e^{-x/2} x^\alpha \, dx < \infty.$$

Thus we have $I_0(m,n) \geq 0$ for every m and n. It follows from (5) that

$$(f\check{e}\phi_\delta)\widehat{\ }(n) \geq h_0^{(\alpha)}\widehat{\phi_\delta}(0)I_0(0,n).$$

Noting that $h_0^{(\alpha)} = 1/\Gamma(\alpha+1), \widehat{\phi_\delta}(0) = 1$ and $I_0(0,n) = \widehat{f\check{e}}(n)$, we have the inequality (4). $\square$

Remark 2. (i) It follows from the above proof that

$$\widehat{f\check{e}}(n) = \sum_{k=0}^{\infty} h_k^{(\alpha)}\gamma(k,0,n;\alpha)\hat{f}(k) \tag{7}$$

for every n, which will be used in the proof of Theorem 1, (ii).

(ii) The assumption $f \in L_\alpha^2$ instead of $f \in L_\alpha^1$ also justifies the identity (6) and thus $I_0(m,n) \geq 0$. For, the inequalities

$$\int_0^\infty |f(x)R_k^{(\alpha)}(x)|e^{-x}x^\alpha\, dx \leq \|f\|_2\ \|R_k^{(\alpha)}\|_2 \leq C_\alpha\|f\|_2 k^{-\alpha/2},$$

hold, where C_α is a constant depending only on α.

2.2. **Wiener type and Paley type theorems for Laguerre expansions.** We have the following theorem which gives our Wiener type theorem and Paley type theorem for the Laguerre polynomial expansions.

Theorem 1. *Let $f \in L_\alpha^1$ and suppose $\hat{f}(n) \geq 0$ for every n.*

(i) *If there exists a constant $\delta > 0$ such that $\int_0^\delta |f(x)|^2 x^\alpha\, dx < \infty$, then $\|f\check{e}\|_2^2 = \int_0^\infty |f(x)e^{-x}|^2 e^{-x}x^\alpha\, dx < \infty$, where $\check{e}(x) = e^{-x}$.*

(ii) *If there exists a constant $\delta > 0$ such that $\operatorname{ess\,sup}_{0\leq x\leq\delta} |f(x)| < \infty$, then $\sum_{n=0}^\infty h_n^{(\alpha)}\hat{f}(n) < \infty$.*

Proof. (i): For $\delta > 0$, let ϕ_δ be the function in Lemma 1. We have by Lemma 2 that

$$\begin{aligned}\sum_{n=0}^\infty h_n^{(\alpha)}\widehat{f\check{e}}(n)^2 &\leq \Gamma(\alpha+1)^2\sum_{n=0}^\infty h_n^{(\alpha)}\{(f\check{e}\phi_\delta)\widehat{\ }(n)\}^2\\ &= \Gamma(\alpha+1)^2\int_0^\infty |f(x)e^{-x}\phi_\delta(x)|^2 e^{-x}x^\alpha\, dx\\ &\leq \Gamma(\alpha+1)^2 \operatorname{ess\,sup}_{0\leq x\leq\delta}|\phi_\delta(x)|^2\cdot\int_0^\delta |f(x)|^2x^\alpha\, dx,\end{aligned}$$

that is, $\|f\check{e}\|_2^2 < \infty$, which completes the proof of (i).

(ii): Let $0 < s < 1$ and $0 < r < 1$. We consider the following convergent double series with nonnegative terms:

$$\sigma(r,s) = \sum_{k=0}^\infty h_k^{(\alpha)}r^k\hat{f}(k)\sum_{n=0}^\infty h_n^{(\alpha)}s^n\gamma(k,0,n;\alpha).$$

It follows from (7) that

$$\sigma(r,s) = \sum_{n=0}^\infty h_n^{(\alpha)}s^n\sum_{k=0}^\infty h_k^{(\alpha)}r^k\gamma(k,0,n;\alpha)\hat{f}(k) \leq \sum_{n=0}^\infty h_n^{(\alpha)}s^n\ \widehat{f\check{e}}(n).$$

By using Lemma 2, we have

$$\begin{aligned}\sigma(r,s) &\le \Gamma(\alpha+1)\sum_{n=0}^{\infty} h_n^{(\alpha)} s^n\ \widehat{f\check{e}\phi_\delta}(n),\\ &= \Gamma(\alpha+1)\sum_{n=0}^{\infty} h_n^{(\alpha)} s^n\ \widehat{f\check{e}\phi_\delta}(n) R_n^{(\alpha)}(0) = \Gamma(\alpha+1) P_s(f\check{e}\phi_\delta; 0).\end{aligned}$$

Therefore we have by (3) that

$$\begin{aligned}\sigma(r,s) &\le \Gamma(\alpha+1)\|P_s^{(\alpha)}(f\check{e}\phi_\delta;\cdot)\|_\infty \le \Gamma(\alpha+1)\left(\frac{2}{1+s}\right)^{\alpha+1}\|f\check{e}\phi_\delta\|_\infty\\ &\le \Gamma(\alpha+1)2^{\alpha+1}\ \operatorname{ess\,sup}_{0\le x\le\delta}|f(x)|.\end{aligned}$$

Letting $r, s \to 1-$, we have that

$$\sum_{k=0}^{\infty} h_k^{(\alpha)}\hat{f}(k)\sum_{n=0}^{\infty} h_n^{(\alpha)}\gamma(k,0,n;\alpha) \le \Gamma(\alpha+1)2^{\alpha+1}\ \operatorname{ess\,sup}_{0\le x\le\delta}|f(x)|,$$

which completes the proof of (ii) since $\sum_{n=0}^{\infty} h_n^{(\alpha)}\gamma(k,0,n;\alpha) = 1$. □

Remark 3. Let $f \in L^1_\alpha$. Suppose that $\sum_{n=0}^{\infty} h_n^{(\alpha)}|\hat{f}(n)| < \infty$. It follows from (1) that the series $\sum_{n=0}^{\infty} h_n^{(\alpha)}\hat{f}(n)R_n^{(\alpha)}(x)e^{-x/2}$ converges absolutely and uniformly to a continuous function on $[0,\infty)$. Since the Poisson integral $P_r^{(\alpha)}(f;\cdot)$ converges to f in L^1_α, we see that

$$f(x) = \sum_{n=0}^{\infty} h_n^{(\alpha)}\hat{f}(n)R_n^{(\alpha)}(x), \quad \text{a.e. } x \in [0,\infty).$$

Therefore, the function f whose values are modified on a set of measure zero is continuous.

Let us prove the following proposition inspired by Theorem 1 (i).

Proposition 1. *Let $f \in L^2_\alpha$, and let N be a positive integer. Suppose that*

$$\int_0^\infty |f(x)e^{-x}|^{2N}x^\alpha\,dx < \infty, \tag{8}$$

and $\hat{f}(n) \ge 0$ for every n. If there exists a constant $\delta > 0$ such that $\int_0^\delta |f(x)|^{2(N+1)}x^\alpha\,dx < \infty$, then $f\check{e} \in L_\alpha^{2(N+1)}$, that is,

$$\int_0^\infty |f(x)e^{-x}|^{2(N+1)}e^{-x}x^\alpha\,dx < \infty. \tag{9}$$

Proof. We put

$$I_N(m,n) = \{(f\check{e})^{N+1}R_m^{(\alpha)}\}\widehat{\ }(n) = \int_0^\infty (f(x)e^{-x})^{N+1}R_m^{(\alpha)}(x)R_n^{(\alpha)}(x)e^{-x}x^\alpha\,dx,$$

and we shall show that $I_N(m,n) \geq 0$ for every m and n. Then we have the desired inequality (9) as follows. Let ϕ_δ be the function in Lemma 1. We have that

$$\begin{aligned}
\{(f\check{e})^{N+1}\phi_\delta\}\widehat{\ }(n) &= \int_0^\infty (f(x)e^{-x})^{N+1}\phi_\delta(x)R_n^{(\alpha)}(x)e^{-x}x^\alpha\,dx \\
&= \sum_{m=0}^\infty h_m^{(\alpha)}\widehat{\phi_\delta}(m)\int_0^\infty (f(x)e^{-x})^{N+1}R_m^{(\alpha)}(x)R_n^{(\alpha)}(x)e^{-x}x^\alpha\,dx, \\
&= \sum_{m=0}^\infty h_m^{(\alpha)}\widehat{\phi_\delta}(m)I_N(m,n). \qquad (10)
\end{aligned}$$

The second equality is justified since

$$\int_0^\infty |(f(x)e^{-x})^{N+1}R_m^{(\alpha)}(x)R_n^{(\alpha)}(x)|e^{-x}x^\alpha\,dx \leq \int_0^\infty |f(x)||f(x)e^{-x}|^N e^{-x}x^\alpha\,dx < \infty$$

by $|R_n^{(\alpha)}(x)e^{-x/2}| \leq 1$ and $f, (f\check{e})^N \in L^2_\alpha$. Since $I_N(m,n) \geq 0$ and $I_N(0,n) = \{(f\check{e})^{N+1}\}\widehat{\ }(n)$, it follows from (10) that

$$\{(f\check{e})^{N+1}\phi_\delta\}\widehat{\ }(n) \geq \frac{1}{\Gamma(\alpha+1)}\{(f\check{e})^{N+1}\}\widehat{\ }(n) \geq 0,$$

which leads to (9).

Let us prove $I_N(m,n) \geq 0$ by induction. Let $N = 1$. Noting $f, f\check{e}R_k^{(\alpha)} \in L^2_\alpha$, we have by the identity (2) that

$$\begin{aligned}
I_1(n,m) &= \sum_{k=0}^\infty \gamma(k,n,m;\alpha)h_k^{(\alpha)}\int_0^\infty f(x)f(x)e^{-x}R_k^{(\alpha)}(x)e^{-x}x^\alpha\,dx, \\
&= \sum_{k=0}^\infty \gamma(k,n,m;\alpha)h_k^{(\alpha)}\sum_{p=0}^\infty \hat{f}(p)\{f\check{e}R_k^{(\alpha)}\}\widehat{\ }(p).
\end{aligned}$$

By Remark 2, (ii), we have $I_0(k,p) = \{f\check{e}R_k^{(\alpha)}\}\widehat{\ }(p) \geq 0$, which leads to $I_1(m,n) \geq 0$.

We also have by (2) that

$$\begin{aligned}
I_N(n,m) &= \sum_{k=0}^\infty \gamma(k,n,m;\alpha)h_k^{(\alpha)}\int_0^\infty f(x)(f(x)e^{-x})^N R_k^{(\alpha)}(x)e^{-x}x^\alpha\,dx, \\
&= \sum_{k=0}^\infty \gamma(k,n,m;\alpha)h_k^{(\alpha)}\sum_{p=0}^\infty \hat{f}(p)I_{N-1}(k,p). \qquad (11)
\end{aligned}$$

The first equality is justified by $f \in L^2_\alpha$ and (8) since

$$\int_0^\infty |f(x)(f(x)e^{-x})^N R_k^{(\alpha)}(x)|e^{-x}x^\alpha\,dx \leq \int_0^\infty |f(x)|\cdot|f(x)e^{-x}|^N e^{x/2}\cdot e^{-x}x^\alpha\,dx.$$

Since $f \in L^2_\alpha$, it is trivial that $\int_0^\infty |f(x)e^{-x}|^2 x^\alpha\,dx < \infty$, with which (8) leads to $\int_0^\infty |f(x)e^{-x}|^{2(N-1)}x^\alpha\,dx < \infty$. By using the assumption $I_{N-1}(k,p) \geq 0$ of induction, we have $I_N(n,m) \geq 0$. $\square$

It may be an interesting problem to find the notion of "solid" space suitable for the Laguerre expansions and extend Theorem 1 or Proposition 1 to such a space.

3. Disk polynomial expansions

In this section, we shall give a Wiener type theorem and a Paley type theorem for the disk polynomial expansions. We shall denote by $\mathbb{D}$ the closed unit disk $\{\ z=x+iy\ ; x^2+y^2\le 1\}$. A function $f(z)$ on $\mathbb{D}$ will be considered as a function $f(x,y)$ of the variables x and y, and a function $f(z,\bar z)$ of the variables z and $\bar z$, where $\bar z = x-iy$, and also a function $f(r,\theta)$ of the variables r and θ, where $z=re^{i\theta}$.

Throughout this section, we suppose that the parameter α satisfies $\alpha>0$. Let m_α be the positive measure of total mass one on $\mathbb{D}$ defined by

$$dm_\alpha(z)=\frac{\alpha+1}{\pi}(1-x^2-y^2)^\alpha dxdy.$$

In this section, for every p with $1\le p\le\infty$, L^p_α stands for the space $L^p(\mathbb{D},m_\alpha)$ and $\|\cdot\|_p$ for $\|\cdot\|_{L^p(\mathbb{D},m_\alpha)}$.

3.1. **Preparations.** In this subsection, we summarize notations and results which will be needed later.

Let m and n be nonnegative integers. The disk polynomials $R^{(\alpha)}_{m,n}(z)$ are defined by

$$R^{(\alpha)}_{m,n}(z)=r^{|m-n|}e^{i(m-n)\theta}R^{(\alpha,|m-n|)}_{m\wedge n}(2r^2-1),\quad z=re^{i\theta},\quad m\wedge n=\min\{m,n\},$$

where $R^{(\alpha,\beta)}_n(x)=P^{(\alpha,\beta)}_n(x)/P^{(\alpha,\beta)}_n(1)$ and $P^{(\alpha,\beta)}_n(x)$ are the Jacobi polynomials given by Rodrigues' formula

$$(1-x)^\alpha(1+x)^\beta P^{(\alpha,\beta)}_n(x)=\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\}.$$

The following inequality holds (cf. [20, (4.1.1) and (7.32.2)]):

$$|R^{(\alpha)}_{m,n}(z)|\le 1,\qquad z\in\mathbb{D}. \tag{12}$$

The system $\{R^{(\alpha)}_{m,n}\}^\infty_{m,n=0}$ is complete orthogonal in L^2_α. The Fourier coefficients $\hat f(m,n)$ of $f\in L^1_\alpha$ for the system $\{R^{(\alpha)}_{m,n}\}^\infty_{m,n=0}$ are defined by

$$\hat f(m,n)=\int_{\mathbb{D}} f(z)\,\overline{R^{(\alpha)}_{m,n}(z)}\,dm_\alpha(z)=\int_{\mathbb{D}} f(z)\,R^{(\alpha)}_{n,m}(z)\,dm_\alpha(z).$$

A function $f\in L^1_\alpha$ on $\mathbb{D}$ is formally expanded as follows:

$$f(z)\sim\sum_{m,n=0}^\infty h^{(\alpha)}_{m,n}\hat f(m,n)R^{(\alpha)}_{m,n}(z),$$

where

$$h^{(\alpha)}_{m,n}=\frac{1}{\|R^{(\alpha)}_{m,n}\|_2^2}=\frac{m+n+\alpha+1}{\alpha+1}\binom{m+\alpha}{m}\binom{n+\alpha}{n}\sim(m+n+1)^{2\alpha+1}.$$

The linearization coefficients for disk polynomials are positive [15, Corollary 5.2]:

$$R^{(\alpha)}_{m,n}(z)R^{(\alpha)}_{k,l}(z)=\sum_{p,q}a(m,n;k,l;p,q)h^{(\alpha)}_{p,q}\overline{R^{(\alpha)}_{p,q}(z)},$$

$$a(m,n;k,l;p,q)=\int_{\mathbb{D}}R^{(\alpha)}_{m,n}(z)R^{(\alpha)}_{k,l}(z)R^{(\alpha)}_{p,q}(z)\,dm_\alpha(z)\ge 0. \tag{13}$$

In the above sum, the pair (p,q) takes such values that $m+k+p=n+l+q$ and $|m+n-k-l|\le p+q\le m+n+k+l$.

Let $1 \le p \le \infty$. For $f \in L^p_\alpha$ and $g \in L^1_\alpha$, the convolution $f * g$ is defined by

$$f * g(\zeta) = \int_{\mathbb{D}} \mathcal{T}_z^{(\alpha)} f(\zeta) g(z)\, dm_\alpha(z), \qquad \zeta \in \mathbb{D},$$

where $\mathcal{T}_z^{(\alpha)}$ is the translation operator for disk polynomials defined by

$$\mathcal{T}_z^{(\alpha)} f(\zeta) = \frac{\alpha}{\alpha+1} \int_{\mathbb{D}} f(\bar{z}\zeta + \sqrt{1-|z|^2}\sqrt{1-|\zeta|^2}\xi)\, \frac{dm_\alpha(\xi)}{1-|\xi|^2}.$$

It is known that

$$\|\mathcal{T}_z^{(\alpha)} f\|_p \le \|f\|_p; \qquad \|f * g\|_p \le \|f\|_p \|g\|_1;$$
$$\widehat{f * g}(m,n) = \hat{f}(m,n)\hat{g}(m,n).$$

We use the following Poisson kernel defined in [11]:

$$\mathcal{P}_s^{(\alpha)}(z) = \sum_{m,n=0}^{\infty} s^{|m-n|+m\wedge n} h_{m,n}^{(\alpha)} R_{m,n}^{(\alpha)}(z), \quad 0 \le s < 1.$$

The Poisson integral of a function $f \in L^p(\mathbb{D}, m_\alpha), 1 \le p \le \infty$ is defined by

$$\mathcal{P}_s^{(\alpha)}(f; z) = (f * \mathcal{P}_s^{(\alpha)})(z) = \sum_{m,n=0}^{\infty} s^{|m-n|+m\wedge n} h_{m,n}^{(\alpha)} \hat{f}(m,n) R_{m,n}^{(\alpha)}(z), \quad z \in \mathbb{D}.$$

We know the following [11, Theorem 5]:

$$\mathcal{P}_s^{(\alpha)}(z) \ge 0, \quad z \in \mathbb{D}\ ; \qquad \int_{\mathbb{D}} \mathcal{P}_s^{(\alpha)}(z)\, dm_\alpha(z) = 1, \quad 0 \le s < 1. \tag{14}$$

Parseval's formula is as follows:

$$\int_{\mathbb{D}} f(z)\, \overline{g(z)}\, dm_\alpha(z) = \sum_{m,n=0}^{\infty} h_{m,n}^{(\alpha)}\ \hat{f}(m,n)\ \overline{\hat{g}(m,n)}$$

for $f, g \in L^2_\alpha$.

We shall use the following result given in [10, Proposition 6.1 and the proof of Theorem 6.3]. It may be difficult to obtain a copy of [10], so we include a proof in the addendum.

Lemma 3 ([10])**.** *Define a differential operator Δ_α by*

$$\Delta_\alpha = 4(1 - z\bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} - 2(\alpha+1) \left(z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right).$$

Then the following (i), (ii) *and* (iii) *hold.*

(i) *The disk polynomials $R_{m,n}^{(\alpha)}$ satisfy*

$$\Delta_\alpha R_{m,n}^{(\alpha)} = -2(\alpha+1) \left(m + n + \frac{2mn}{\alpha+1} \right) R_{m,n}^{(\alpha)}. \tag{15}$$

(ii) *For $f, g \in C^2(\mathbb{D})$,*

$$\int_{\mathbb{D}} \Delta_\alpha f(z) \overline{g(z)}\, dm_\alpha(z) = \int_{\mathbb{D}} f(z) \overline{\Delta_\alpha g(z)}\, dm_\alpha(z). \tag{16}$$

(iii) *Let $f \in C^\infty(\mathbb{D})$. For every positive integer k, there exists a positive constant C such that*

$$|\hat{f}(m,n)| \le C(m+n+1)^{-k}, \qquad m, n = 0, 1, 2, \ldots. \tag{17}$$

We shall construct a function having properties similar to the function ϕ_δ in Lemma 1. For a and λ with $0 < a < 1$ and $0 < \lambda < \pi$, we use the following notation:

$$\bar{S}(a, \lambda) = \{\ z = se^{i\phi}\ : a \le s \le 1,\ |\phi| \le \lambda\ \}. \tag{18}$$

Lemma 4. *For a with $0 < a < 1$, put*

$$b(a) = \frac{1}{2}(a + \sqrt{2 - a^2}), \qquad \lambda(a) = \pi \frac{\sqrt{1-a^2}}{a} \frac{\sqrt{1-b(a)^2}}{b(a)}.$$

Suppose $1/\sqrt{2} < a < 1$. Then there exists a function ψ_a on $\mathbb{D}$ such that $\operatorname{supp} \psi_a \subset \bar{S}(a, \lambda(a))$, $\widehat{\psi_a}(0,0) = 1$, $\widehat{\psi_a}(m,n) \ge 0$ for every m and n, and $\widehat{\psi_a}(m,n) = O((m+n)^{-k})$ as $m, n \to +\infty$ for any positive integer k.

Proof. We note first that (i) $a < b(a) < 1$ for $0 < a < 1$; (ii) $0 \le \sqrt{1-r^2}/r < 1$ for $1/\sqrt{2} < r \le 1$; (iii) $\sqrt{1-r^2}/r \downarrow +0$ as $r \to 1-$; (iv) $0 < \lambda(a) < \pi$ for $0 < a < 1$.

Let $1/\sqrt{2} < a < 1$. We choose a function $h_a \in C^\infty(\mathbb{D})$ such that $h_a \ge 0$, $\operatorname{supp} h_a \subset \bar{S}(b(a), \lambda(a)/4)$ and

$$\widehat{h_a}(0,0) = \int_{\mathbb{D}} h_a(z)\, dm_\alpha(z) = 1.$$

Put $\check{h}_a(z) = h_a(\bar{z})$. Then $\check{h}_a$ has the same properties as h_a. Let ψ_a be a function on $\mathbb{D}$ such that

$$\psi_a(\zeta) = h_a * \check{h}_a(\zeta) = \int_{\bar{S}(b(a),\lambda(a)/4)} \mathcal{T}_z^{(\alpha)} h_a(\zeta) \check{h}_a(z)\, dm_\alpha(z), \qquad \zeta \in \mathbb{D}.$$

We show first that $\operatorname{supp} \psi_a \subset \bar{S}(a, \lambda(a))$. It is enough to show that for $z \in \bar{S}(b(a), \lambda(a)/4)$ and $\zeta \notin \bar{S}(a, \lambda(a))$

$$\mathcal{T}_z^{(\alpha)} h_a(\zeta) = \int_{\mathbb{D}} h_a \left(\bar{z}\zeta + \sqrt{1-|z|^2}\sqrt{1-|\zeta|^2}\xi \right) \frac{dm_\alpha(\xi)}{1-|\xi|^2} = 0,$$

which will follow from

$$\bar{z}\zeta + \sqrt{1-|z|^2}\sqrt{1-|\zeta|^2}\xi\ \notin\ \bar{S}(b(a), \lambda(a)/4) \tag{19}$$

for $\xi \in \mathbb{D}$, $z \in \bar{S}(b(a), \lambda(a)/4)$ and $\zeta \notin \bar{S}(a, \lambda(a))$. We show this. We write $z, \zeta \in \mathbb{D}$ by using the polar coordinates as $z = se^{i\phi}, -\pi < s \le \pi, 0 \le s \le 1$ and $\zeta = re^{i\theta}, -\pi < \theta \le \pi, 0 \le r \le 1$. Assume $r < a$. Then for $\xi \in \mathbb{D}$ and $z \in \bar{S}(b(a), \lambda(a)/4)$, we have by the definition of $b(a)$ that

$$\left| \bar{z}\zeta + \sqrt{1-|z|^2}\sqrt{1-|\zeta|^2}\xi \right| < a + \sqrt{1-b(a)^2} = b(a). \tag{20}$$

Next we suppose $a \le r \le 1$ and $|\theta| > \lambda(a)$. Let $\xi \in \mathbb{D}$ and $z \in \bar{S}(b(a), \lambda(a)/4)$. We define ω by the equation

$$\sin \omega = \frac{\sqrt{1-s^2}\sqrt{1-r^2}}{sr}, \qquad 0 \le \omega < \frac{\pi}{2}.$$

It follows that $\omega \le \pi\sqrt{1-s^2}\sqrt{1-r^2}/(2sr) \le \lambda(a)/2$. If $\lambda(a) < \theta \le \pi$, then

$$\begin{aligned} \frac{1}{4}\lambda(a) \le \lambda(a) - \frac{1}{4}\lambda(a) - \omega < \arg \left(\bar{z}\zeta + \sqrt{1-|z|^2}\sqrt{1-|\zeta|^2}\xi \right) \\ \le \pi + \frac{1}{4}\lambda(a) + \omega < \frac{7}{4}\pi < 2\pi - \frac{1}{4}\lambda(a). \end{aligned} \tag{21}$$

For $-\pi < \theta < -\lambda(a)$, we have

$$-2\pi + \frac{1}{4}\lambda(a) < \arg\left(\bar{z}\zeta + \sqrt{1-|z|^2}\sqrt{1-|\zeta|^2}\xi\right) < -\frac{1}{4}\lambda(a) \tag{22}$$

in the same way. By combining (20), (21) and (22), we have (19), which shows $\operatorname{supp}\psi_a \subset \bar{S}(a,\lambda(a))$.

Since $\widehat{\psi_a}(m,n) = |\widehat{h_a}(m,n)|^2$, it follows that $\widehat{\psi_a}(0,0) = 1$ and $\widehat{\psi_a}(m,n) \geq 0$ for every m and n. Also, Lemma 3 leads us to $\widehat{\psi_a}(m,n) = O((m+n)^{-k})$ as $m,n \to +\infty$ for any positive integer k. □

Remark 4. We easily see that the function ψ_a is continuous on $\mathbb{D}$. It follows from (12) that the series $\sum_{m,n=0}^{\infty} h_{m,n}^{(\alpha)}\widehat{\psi_a}(m,n)R_{n,m}^{(\alpha)}(z)$ converges uniformly to a continuous function on $\mathbb{D}$. We know that the Poisson integral $\mathcal{P}_s^{(\alpha)}(\psi_a;\cdot)$ converges to ψ_a in $L_\alpha^p, 1 \leq p < \infty$ ([11, Corollary 6]). From these, we see that

$$\psi_a(z) = \sum_{m,n=0}^{\infty} h_{m,n}^{(\alpha)}\widehat{\psi_a}(m,n)R_{n,m}^{(\alpha)}(z),$$

where the series converges absolutely and uniformly on $\mathbb{D}$. Moreover, it is not hard to prove $\psi_a \in C^\infty(\mathbb{D})$ by using (15).

Lemma 5. *For a with $1/\sqrt{2} < a < 1$, let ψ_a be the function in Lemma 4. Suppose that $f \in L^1(\mathbb{D}, m_\alpha)$ and $\hat{f}(m,n) \geq 0$ for every m and n. Then,*

$$\hat{f}(m,n) \leq \widehat{f\psi_a}(m,n) \tag{23}$$

for every m and n.

Proof. Since $f \in L^1(\mathbb{D}, m_\alpha)$ and the expansion of ψ_a converges boundedly on $\mathbb{D}$, it follows that

$$(f\psi_a)\widehat{\ }(m,n) = \sum_{k,l=0}^{\infty} h_{k,l}^{(\alpha)}\widehat{\psi_a}(k,l)\int_{\mathbb{D}} f(z)R_{k,l}^{(\alpha)}(z)R_{n,m}^{(\alpha)}(z)\,dm_\alpha(z).$$

By (13), we have

$$\int_{\mathbb{D}} f(z)R_{k,l}^{(\alpha)}(z)R_{n,m}^{(\alpha)}(z)\,dm_\alpha(z) = \sum_{p,q} a(k,l;n,m;p,q)h_{p,q}^{(\alpha)}\hat{f}(p,q).$$

Since all the terms appearing in the sums are positive, it follows that

$$(f\psi_a)\widehat{\ }(m,n) \geq h_{0,0}^{(\alpha)}\widehat{\psi_a}(0,0)a(0,0;n,m;m,n)h_{m,n}^{(\alpha)}\hat{f}(m,n).$$

We note that $h_{0,0}^{(\alpha)} = 1$, $\widehat{\psi_a}(0,0) = 1$ and $a(0,0;n,m;m,n) = h_{m,n}^{(\alpha)\,-1}$, which completes the proof. □

3.2. **Wiener type and Paley type theorems.** Wiener type and Paley type theorems for the disk polynomial expansions are as follows.

Theorem 2. *Let $f \in L^1(\mathbb{D}, m_\alpha)$ and $\hat{f}(m,n) \geq 0$ for every m and n.*

(i) *If there exist constants a_0 and λ_0 with $0 < a_0 < 1$, $0 < \lambda_0 < \pi$ such that $\int_{\bar{S}(a_0,\lambda_0)} |f(z)|^2\,dm_\alpha(z) < \infty$, then $\|f\|_2^2 = \int_{\mathbb{D}} |f(z)|^2\,dm_\alpha(z) < \infty$, where $\bar{S}(a_0,\lambda_0)$ is defined by* (18).

(ii) *If there exist constants a_0 and λ_0 with $0 < a_0 < 1$, $0 < \lambda_0 < \pi$ such that $\operatorname{ess\,sup}_{z\in\bar{S}(a_0,\lambda_0)} |f(z)| < \infty$, then $\sum_{m,n=0}^{\infty} h_{m,n}^{(\alpha)}\hat{f}(m,n) < \infty$.*

Proof. We choose a such that $a_0 < a < 1$, $1/\sqrt{2} < a$ and $\lambda(a) < \lambda_0$, and let ψ_a be the function in Lemma 4. By Lemma 5 and $\bar{S}(a, \lambda(a)) \subset \bar{S}(a_0, \lambda_0)$, we have

$$\begin{aligned}\sum_{m,n=0}^{\infty} h_{m,n}^{(\alpha)}\{\hat{f}(m,n)\}^2 &\leq \sum_{m,n=0}^{\infty} h_{m,n}^{(\alpha)}\{(f\psi_a)\widehat{\ }(m,n)\}^2 \\ &= \int_{\mathbb{D}} |f(z)\psi_a(z)|^2\, dm_\alpha(z) \\ &\leq \max_{z\in\bar{S}(a,\lambda(a))} |\psi_a(z)|^2 \cdot \int_{\bar{S}(a_0,\lambda_0)} |f(z)|^2\, dm_a(z).\end{aligned}$$

This means $\|f\|_2^2 < \infty$, which completes the proof of (i).

Let $0 < s < 1$. By Lemma 5 and $R_{m,n}^{(\alpha)}(1) = 1$, we have

$$\begin{aligned}\sum_{m,n=0}^{\infty} s^{|m-n|+m\wedge n} h_{m,n}^{(\alpha)}\hat{f}(m,n) &\leq \sum_{m,n=0}^{\infty} s^{|m-n|+m\wedge n} h_{m,n}^{(\alpha)}\widehat{f\psi_a}(m,n) R_{m,n}^{(\alpha)}(1), \\ &= \mathcal{P}_s^{(\alpha)}(f\psi_a; 1) \leq \|\mathcal{P}_s^{(\alpha)}(f\psi_a; \cdot)\|_\infty.\end{aligned}$$

By (14), we see that $\|\mathcal{P}_s^{(\alpha)}(f\psi_a;\cdot)\|_\infty \leq \|f\psi_a\|_\infty$, which implies

$$\sum_{m,n=0}^{\infty} s^{|m-n|+m\wedge n} h_{m,n}^{(\alpha)}\hat{f}(m,n) \leq \max_{z\in\bar{S}(a,\lambda(a))} |\psi_a(z)| \cdot \operatorname*{ess\,sup}_{z\in\bar{S}(a_0,\lambda_0)} |f(z)|.$$

Letting $s \to 1-$, we complete the proof of (ii). $\square$

Remark 5. Let $f \in L^1(\mathbb{D}, m_\alpha)$ be a function in Theorem 2 (ii). Then we can modify the values of f on a set of measure 0 with respect to dm_α so that f is continuous and

$$f(z) = \sum_{m,n=0}^{\infty} h_{m,n}^{(\alpha)}\hat{f}(m,n) R_{m,n}^{(\alpha)}(z),$$

the series converges absolutely and uniformly on $\mathbb{D}$.

We can obtain the analogue of Theorem 2 (i) for $L^{2N}(\mathbb{D}, m_\alpha), N = 1, 2, 3, \ldots,$ that is, we have the following.

Proposition 2. *Let $f \in L_\alpha^1$ and $\hat{f}(m,n) \geq 0$ for every m and n. If there exist constants a_0 and λ_0 with $0 < a_0 < 1$, $0 < \lambda_0 < \pi$ such that $\int_{\bar{S}(a_0,\lambda_0)} |f(z)|^{2N}\, dm_\alpha(z) < \infty$, then $\int_{\mathbb{D}} |f(z)|^{2N}\, dm_\alpha(z) < \infty$.*

Proof. We shall show that if $h \in L_\alpha^1$ and $g \in L_\alpha^{2N}$ satisfy $|\hat{h}(m,n)| \leq \hat{g}(m,n)$ for every m and n, then $h \in L_\alpha^{2N}$. Then taking $h = f$ and $g = f\psi_a$, we have the proposition owing to Lemma 5. To show $h \in L_\alpha^{2N}$, we prove that every $\widehat{h^N}(m,n)$ exists and $|\widehat{h^N}(m,n)| \leq \widehat{g^N}(m,n)$. We show this by induction. The case $N = 1$ is clear. Assume that $h \in L_\alpha^1$, $g \in L_\alpha^{2(N+1)}$ and $|\hat{h}(m,n)| \leq \hat{g}(m,n)$ for every m and n. It follows from the assumption of induction that $h^N \in L_\alpha^2$. By Parseval's identity and (13), we have

$$
\begin{aligned}
(h^{N+1})\widehat{\ }(m,n) &= \int_{\mathbb{D}} h^N(z)h(z)\overline{R^{(\alpha)}_{m,n}(z)}\,dm_\alpha(z),\\
&= \sum_{k,l=0}^{\infty} \widehat{h^N}(k,l)\overline{(\bar{h}R^{(\alpha)}_{m,n})\widehat{\ }(k,l)},\\
&= \sum_{k,l=0}^{\infty} \widehat{h^N}(k,l)\sum_{p,q} a(m,n;l,k;p,q)h^{(\alpha)}_{p,q}\hat{h}(p,q).
\end{aligned}
$$

In the same way, we have the above identity with g instead of h. Therefore, the assumption of induction completes the proof. $\square$

We can extend Proposition 2 to a larger class of solid spaces than L^{2N}_α. A subspace $X \subset L^1_\alpha$ is called solid if $f, g \in L^1_\alpha$, $|\hat{f}(m,n)| \le \hat{g}(m,n)$ for every m and n, and $g \in X$ imply that $f \in X$. Let X_{loc} be the space of functions $f \in L^1_\alpha$ satisfying the condition that there exist positive constants a_0 and λ_0 with $0 < a < 1, 0 < \lambda_0 < \pi$ such that $f\psi \in X$ for any $\psi \in C^\infty$ with supp $\psi \subset \bar{S}(a_0, \lambda_0)$. We denote by $\mathbb{P}$ the space of functions $f \in L^1_\alpha$ satisfying $\hat{f}(m,n) \ge 0$ for every m and n. Then, by Lemma 5 we easily obtain the following result: *If X is a solid space, then $X_{loc} \cap \mathbb{P} = X \cap \mathbb{P}$.* This is an extension of Proposition 2 since the spaces $L^{2N}_\alpha, N = 1, 2, 3, \dots$ are solid, which was already proved in the proof of the proposition. This extension is the disk polynomial analogue of the theorem on the Jacobi polynomials obtained by Mhaskar and Tikhonov [17, Theorem 3.1].

Addendum

For readers' convenience, we shall give a proof of Lemma 3 by following the lines of Heyer and Koshi [10].

Let us give a proof of (i) of the lemma. We treat the case $m \ge n$. In this case, we have

$$R^{(\alpha)}_{m,n}(z) = z^{m-n}R^{(\alpha,m-n)}_n(2z\bar{z}-1), \qquad z \in \mathbb{D}.$$

Substituting $u = 2z\bar{z} - 1$, we have

$$
\begin{aligned}
\Delta_\alpha R^{(\alpha)}_{m,n}(z) = z^{m-n}\Bigg\{&4(1-u^2)\frac{d^2}{du^2}R^{(\alpha,m-n)}_n(u)\\
&+ 4\left((m-n-\alpha) - (\alpha+m-n+2)u\right)\frac{d}{du}R^{(\alpha,m-n)}_n(u)\\
&- 2(\alpha+1)(m-n)R^{(\alpha,m-n)}_n(u)\Bigg\}.
\end{aligned}
$$

Since $-2(\alpha+1)(m-n) = 4n(n+\alpha+(m-n)+1) - 2(\alpha+1)(m+n+2mn/(\alpha+1))$, the differential equation [20, (4.2.1)] leads to the identity (15). The case $m \le n$ will be done similarly. We completes the proof of (i) of the lemma.

We state a proof of (ii) of the lemma. For $0 < \epsilon < 1$, we put $\mathbb{D}_\epsilon = \{\, z \ : |z| \le 1 - \epsilon\}$. Since

$$(1-x^2-y^2)^\alpha\Delta_\alpha f = \frac{\partial}{\partial x}(1-x^2-y^2)^{\alpha+1}\frac{\partial f}{\partial x} + \frac{\partial}{\partial y}(1-x^2-y^2)^{\alpha+1}\frac{\partial f}{\partial y},$$

it follows from Green's formula that

$$
\begin{aligned}
&\int_{\mathbb{D}_\epsilon} (\Delta_\alpha f)\bar{g}\, dm_\alpha \\
&= \frac{\alpha+1}{\pi} \int_{\mathbb{D}_\epsilon} \left\{ \frac{\partial}{\partial x}\left((1-x^2-y^2)^{\alpha+1} \frac{\partial f}{\partial x}\bar{g} \right) + \frac{\partial}{\partial y}\left((1-x^2-y^2)^{\alpha+1} \frac{\partial f}{\partial y}\bar{g} \right) \right\} dxdy \\
&\qquad\qquad - \frac{\alpha+1}{\pi} \int_{\mathbb{D}_\epsilon} \left(\frac{\partial f}{\partial x}\frac{\partial \bar{g}}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \bar{g}}{\partial y} \right) (1-x^2-y^2)^{\alpha+1}\, dxdy, \\
&= \frac{\alpha+1}{\pi} \int_{\partial\mathbb{D}_\epsilon} (1-x^2-y^2)^{\alpha+1} \left(\frac{\partial f}{\partial x}\bar{g}\, dy - \frac{\partial f}{\partial y}\bar{g}\, dx \right) \\
&\qquad\qquad - \frac{\alpha+1}{\pi} \int_{\mathbb{D}_\epsilon} \left(\frac{\partial f}{\partial x}\frac{\partial \bar{g}}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \bar{g}}{\partial y} \right) (1-x^2-y^2)^{\alpha+1}\, dxdy.
\end{aligned}
$$

The first contour integral satisfies

$$\int_{\partial\mathbb{D}_\epsilon} (1-x^2-y^2)^{\alpha+1} \left(\frac{\partial f}{\partial x}\bar{g}\, dy - \frac{\partial f}{\partial y}\bar{g}\, dx \right) = O(\epsilon^{\alpha+1}), \quad \epsilon \to 1-.$$

Thus we have

$$\int_{\mathbb{D}} (\Delta_\alpha f)\bar{g}\, dm_\alpha = -\frac{\alpha+1}{\pi} \int_{\mathbb{D}} \left(\frac{\partial f}{\partial x}\frac{\partial \bar{g}}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \bar{g}}{\partial y} \right) (1-x^2-y^2)^{\alpha+1}\, dxdy.$$

In this equality, we replace the functions f and $\bar{g}$ by $\bar{g}$ and $\bar{f}$, respectively. Then we see that $\int_{\mathbb{D}} (\Delta_\alpha f)\bar{g}\, dm_\alpha = \int_{\mathbb{D}} f\overline{\Delta_\alpha g}\, dm_\alpha$, which is (16). The proof of (ii) of the lemma is complete.

A proof of (iii) of the lemma is as follows. Let $f \in C^\infty(\mathbb{D})$, and let k be a positive integer. We choose a positive integer r such that $k \le r + \alpha + 1/2$. By (i) and (ii) of the lemma, we have that

$$
\begin{aligned}
(-2)^r(\alpha+1)^r \left(m+n+\frac{2mn}{\alpha+1} \right)^r \hat{f}(m,n) &= \int_{\mathbb{D}} f(z)\overline{(\Delta_\alpha)^r R^{(\alpha)}_{m,n}(z)}\, dm_\alpha(z), \\
&= \int_{\mathbb{D}} (\Delta_\alpha)^r f(z)\overline{R^{(\alpha)}_{m,n}(z)}\, dm_\alpha(z),
\end{aligned}
$$

and that

$$\left| \int_{\mathbb{D}} (\Delta_\alpha)^r f(z)\overline{R^{(\alpha)}_{m,n}(z)}\, dm_\alpha(z) \right| \le \|R^{(\alpha)}_{m,n}\|_2 \|(\Delta_\alpha)^r f\|_2.$$

Since $\|R^{(\alpha)}_{m,n}\|_2 \le C(m+n+1)^{-\alpha-1/2}$, it follows that

$$|\hat{f}(m,n)| \le C(m+n+1)^{-r-\alpha-1/2} \le C(m+n+1)^{-k}$$

with a positive constant C independent of m and n, which completes the proof of (iii) of the lemma.

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Mathematics Section, Division of Innovative Technology and Science, Graduate School of Natural Science and Technology, Kanazawa University Kanazawa 920-1192, Japan

E-mail address: `kanjin@staff.kanazawa-u.ac.jp`