Moment stability in mean square of stochastic delay differential equation

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Peng Xue, Shigeru Yamamoto and Yosuke Ikei

Abstract— **In this paper, we derive a moment stability region in terms of coefficient parameters for a stochastic delay differential equation. Such a stochastic delay equation with both time delay and random effects is an essential model of control systems. As a main result, a fundamental stability problem is solved by delay-dependent stochastic analysis. We adopt the domain-subdivision approach and use Ito's formula in the analysis. For a given time delay, the stability of the stochastic delay differential equation is studied with variable power of noise. It is also shown that an unstable stochastic delay system become stable by an appropriate power of noise. The main results are illustrated by several numerical solutions of the stochastic delay model.**

Index Terms— **Time delay, Stochastic, Stabilization.**

I. INTRODUCTION

Analysis and control for stochastic delay differential systems have been intensively studied in the control community during the past decade. Among them, the study of the fluctuations in the center of pressure during quiet standing shows two remarkable properties in human motor control mechanism, the existence of time delays and random fluctuations [1][2]. Generally, time delay and noise make more difficult to control of machine systems. Therefore, it is important to give an answer of how humans can compensate the large time delay.

Complex fluctuations are ubiquitous for real systems in nature. The Ito's formula is widely used in stability analysis of stochastic differential equations. The system can be stabilized by noise in the sense of probability one but not in the p-th moment stability. On the other hand, time delay is also negative factor for the stability of control system. The stochastic differential delay equations represent a relatively new field of the qualitative theory of differential equations [3]. Even for a simple linear stochastic differential delay equation

$$\mathrm{d}x(t) = (ax(t)+bx(t-\tau))\mathrm{d}t+(dx(t)+cx(t-\tau))\mathrm{d}w, \quad (1)$$

stability analysis is not derived yet, where $\tau > 0$ denotes time delay and $w(t) \in R$ is a standard Wiener process. Our interesting focuses on the relationship between noise and time delay in the sense of asymptotic p-th moment stability with $d = 0$.

Peng Xue and Yosuke Ikei are with Graduate School of Natural Science and Technology, Kanazawa University, Kakuma-machi, Kanazawa, Ishikawa, 920-1192, JAPAN. {xuepeng,ikei}@moccos.ec.t.kanazawa-u.ac.jp.

Shigeru Yamamoto is with Faculty of Electrical and Computer Engineering, Kanazawa University, Kakuma-machi, Kanazawa, Ishikawa, 920-1192, JAPAN. shigeru@t.kanazawa-u.ac.jp.

Several conditions for determining the asymptotic stability regions in the controller parameter space have been studied[4][5]. These approaches are based on the Lyapunov theory and LMI (Linear Matrix Inequality)[6] [7]. An alternative approach to the problem of stabilizing plants with time delay was described in [8]. By using the classical domain subdivision (D-subdivision) method, simple and efficient computational methods are presented for determining the asymptotic stability and computing D-stability region in the parameter space are presented.

For a given time delay, the stability region for a stochastic delay differential equation is studied in this paper. In Section II, the mathematical preliminaries, Ito's formula and D-subdivision, will be reviewed in Section II. In Section III, our main result will be derived. Section IV illustrates several examples.

II. MATHEMATICAL PRELIMINARIES

We denote a stationary Gaussian white noise process by $\xi(t)$ and assume that the expectation of $\xi(t)$ satisfies $E(\xi(t)) = 0$. Our approach will be taken in this paper is based on the Ito's formula and D-subdivision which are summarized in the following subsections.

A. Ito's formula

Definition 1 (stochastic integral): A stochastic integral for a stochastic process x is defined by

$$x(t) = x(0) + \int_0^t u(x(s), w)\mathrm{d}s + \int_0^t v(x(s), w)\mathrm{d}w, \quad (2)$$

where v satisfies the integrability condition

$$E\left\{\int_0^T v^2(x(t), w)dt\right\} < +\infty. \quad (3)$$

Equivalently, the stochastic integral (2) is also written as a stochastic differential equation of the form

$$\mathrm{d}x = u\mathrm{d}t + v\mathrm{d}w. \quad (4)$$

In the following, we state the Ito's formula which is the fundamental theorem for computing Ito integrals[8].

Lemma 1 (Ito formula): Let x satisfies a stochastic differential equation (4) and $g(t, x)$ be a twice continuously differentiable function on $[0, T] \times R$. Then the stochastic process $y = g(t, x)$ has a stochastic integral with

$$\mathrm{d}y = \frac{\partial}{\partial t}g(t, x)\mathrm{d}t + \frac{\partial}{\partial x}g(t, x)\mathrm{d}x + \frac{1}{2}\frac{\partial^2}{\partial x^2}g(t, x)(\mathrm{d}x)^2 \quad (5)$$

where $(dx)^2$ is given by the following rules

$$(\mathrm{d}t)^2 = \mathrm{d}t\mathrm{d}w = \mathrm{d}w\mathrm{d}t = 0 \quad \text{and} \quad (\mathrm{d}w)^2 = \mathrm{d}t. \quad (6)$$

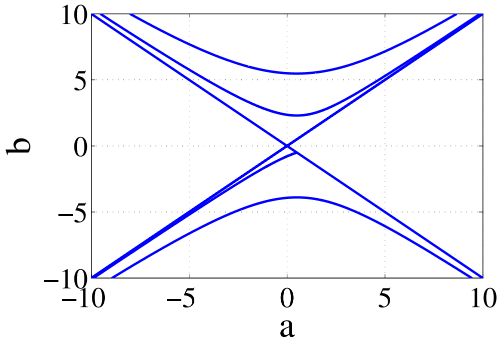

Fig. 1. D-Subdivision for the transcendental function (9) assuming $\tau = 2$

Next, we compute the integral form of (5). By (6) we have

$$(\mathrm{d}x)^2 = (u\mathrm{d}t + v\mathrm{d}w)(u\mathrm{d}t + v\mathrm{d}w) = v^2\mathrm{d}t. \quad (7)$$

Therefore, (5) is written in the form

$$\mathrm{d}y = \left[\frac{\partial}{\partial t}g(t,x) + \frac{\partial}{\partial x}g(t,x)u + \frac{1}{2}\frac{\partial^2}{\partial x^2}g(t,x)v^2\right]\mathrm{d}t + \left[\frac{\partial g}{\partial x}(t,x)v(t)\right]\mathrm{d}w. \quad (8)$$

B. D-subdivision

Given a transcendental function like, for example,

$$\lambda + a + be^{-\lambda\tau} = 0. \quad (9)$$

The so-called D-subdivision method is able to determine the number of roots having positive real part in accordance with the value of its parameters[9]. It is possible because the zeros of a transcendental function are continuous functions of their parameters. D-subdivision divides the space of coefficients into regions by hypersurfaces, which corresponds to quasi-polynomial parameters having at least one zero on the imaginary axis. For continuous variation of the trascendental function parameters, the number of zeros of the transcendental function, defined by the points of this region. Finally in order to clarify how the number of roots with positive real parts changes as some boundary of the D-subdivision is crossed, the differential of the real part of the root is computed, and the decrease or increase of the number of p-zeros is determined from its algebraic sign.

III. EXPONENTIAL STABILITY OF SCALAR LINEAR DIFFERENTIAL DELAY EQUATION

Consider the following scalar linear stochastic differential delay equation with Gaussian white noise

$$\begin{aligned}&\mathrm{d}x(t) = (ax(t) + bx(t-\tau))\,\mathrm{d}t + cx(t-\tau)\mathrm{d}w,\\ &x(0+h) = x_0, h \in [-\tau, 0],\end{aligned} \quad (10)$$

where $\tau > 0$ is a constant delay.

Our interest is the stability of the trivial solution of the stochastic differential delay equation(10)

A stochastic process $x(t)$ is called a solution of the stochastic differential equation (10) when it satisfies, with probability one, the integral equation

$$\begin{aligned}x(t) =& x(0) + a\int_0^t x(s)\mathrm{d}s + b\int_0^t x(s-\tau)\mathrm{d}s\\ &+ c\int_0^t x(s-\tau)\mathrm{d}w\end{aligned} \quad (11)$$

where the third integral is Ito's stochastic integral.

We introduce the following definition of stability for stochastic differential delay equations.

Definition 2 (asymptotically p-th moment stable): The trivial solution of (10) is called to be asymptotically p-th moment stable if for any initial function ϕ,

$$\lim_{t\to\infty} E\{\|x(t,\phi)\|^p\} = 0. \quad (12)$$

In particular, it is said to be mean square stable when $p = 2$.

Using the idea of mean square stable, we study the behavior of the second moment stability of a kind of the stochastic diferential delay equation. Our main result is given as the following.

Theorem 1: The stochastic delay equation (10) with $2a + 2b + c^2 \neq 0$ is mean square stable if

$$a < \overline{a} \text{ and } \underline{b} < b < \overline{b}, \quad (13)$$

where for a minimal positive solution ω_0 of $c^2\sin(\omega_0\tau) = \omega_0$,

$$\overline{a} = \begin{cases}\dfrac{c^2}{2} + \dfrac{1}{\tau} & \text{for } c \in [0, \dfrac{1}{\sqrt{\tau}}]\\ \dfrac{c^2}{2} + \omega_0\dfrac{\cos(\omega_0\tau)}{\sin(\omega_0\tau)} & \text{for } c \in (\dfrac{1}{\sqrt{\tau}}, +\infty),\end{cases} \quad (14)$$

$$\overline{b} = -a - \frac{c^2}{2},$$

and for any solution ω of $\omega\cot(\omega\tau) = a - \dfrac{c^2}{2}$, by using $\omega \in (0, \pi/\tau)$ for $c \in [0, 1/\sqrt{\tau}]$, or $\omega \in (\omega_0, \pi/\tau)$ for $c \in (1/\sqrt{\tau}, +\infty)$, we define

$$\underline{b} = -c^2\cos(\omega\tau) - \frac{\omega}{\sin(\omega\tau)}.$$

Proof: By using the Ito's differential rule, we obtain

$$\begin{aligned}\mathrm{d}x^2(t) =& (2ax^2(t) + 2bx(t)x(t-\tau) + c^2x^2(t-\tau))\mathrm{d}t\\ &+ 2cx(t)x(t-\tau)\mathrm{d}w.\end{aligned} \quad (15)$$

Integrating from 0 to t, taking the mathematical expectation and differentiating with respect to t, since $E(\int_0^t 2cx(t)x(t-\tau)\mathrm{d}w) = 0$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(x^2(t)) =& 2aE(x^2(t)) + 2bE(x(t)x(t-\tau))\\ &+ c^2E(x^2(t-\tau)).\end{aligned} \quad (16)$$

By introducing the notation $K(t,s) := E\,[x(t)x(s)]$, from equation (16) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}K(t,t) =& 2aK(t,t) + 2bK(t,t-\tau)\\ &+ c^2K(t-\tau, t-\tau).\end{aligned} \quad (17)$$

We assume the existence of the steady state solution K^* of (17),

$$\begin{aligned}\lim_{t\to\infty} K(t,t) &= \lim_{t\to\infty} K(t,t-\tau)\\ &= \lim_{t\to\infty} K(t-\tau,t-\tau) := K^* \end{aligned} \tag{18}$$

which satisfies $0 = 2aK^* + 2bK^* + c^2K^*$. Hence, if $2a + 2b + c^2 \neq 0$, then

$$\lim_{t\to\infty} E\left[x^2(t)\right] = K^* = 0. \tag{19}$$

For a solution $K(t,s) = K(0,0)e^{\lambda t}e^{\lambda s}$ to (17)[2], we obtain the characteristic function as

$$2\lambda = 2a + 2be^{-\lambda\tau} + c^2e^{-2\lambda\tau}. \tag{20}$$

The trivial solution of (17) is exponentially asymptotically stable in the Lyapunov sense if and only if all the infinitely many characteristic roots of the characteristic equation (20) have negative real parts. Clearly, there exists a pure imaginary characteristic root $\lambda = j\omega$, $\omega > 0$ at the limit of asymptotic stability. Substitute this root into (20) and separate the real and imaginary parts of the resulting complex equation

$$\begin{cases}\omega + b\sin(\omega\tau) + c^2\sin(\omega\tau)\cos(\omega\tau) = 0\\ a + b\cos(\omega\tau) + c^2\cos^2(\omega\tau) - \dfrac{c^2}{2} = 0.\end{cases} \tag{21}$$

They are equivalent to

$$\text{for } \omega = 0: \qquad a + b + \frac{c^2}{2} = 0, \tag{22}$$

$$\text{for } \omega \neq 0: \qquad \begin{cases} a = \dfrac{c^2}{2} + \omega\dfrac{\cos(\omega\tau)}{\sin(\omega\tau)}\\ b = -c^2\cos(\omega\tau) - \dfrac{\omega}{\sin(\omega\tau)}.\end{cases} \tag{23}$$

The equations in parametric form (21) identify all the other D-subdivision boundaries. To be precise, there exists one boundary for any of the following interval of ω:

$$(0,\pi/\tau), (\pi/\tau, 2\pi/\tau), (2\pi/\tau, 3\pi/\tau), \cdots. \tag{24}$$

In particular, we focus on the interval $0 \leq \omega < \pi/\tau$. We shall show how zeros rises as following. By (21), the stable boundaries are illustrated in Fig. 2. Let $\omega \to 0$, the asymptotical intersection is obtained as $(1/\tau + c^2/2, -1/\tau - c^2/2)$. The parameters plant is divided into two parts by the curves C_1 and C_2. Taking the left part as Γ_1, the right part as Γ_2, see Fig. 2. To study how the $\text{Re}(\lambda)$ invariance when parametric invaries from Γ_1 to Γ_2, the differential of $\text{dRe}(\lambda)/\text{d}b$ is computed for $\omega = 0$ and $\omega \in (0,\pi/\tau)$ as following.

From (20), we obtain

$$\frac{\text{d}}{\text{d}b}\text{Re}(\lambda) = \text{Re}\frac{e^{-\lambda\tau}}{1 + b\tau e^{-\lambda\tau} + c^2\tau e^{-2\lambda\tau}}. \tag{25}$$

To consider the boundaries, by substituting $\lambda = j\omega$ into (25) and defining function

$$\begin{aligned}\Phi(\omega) = &(1 + b\tau\cos(\omega\tau) + c^2\tau\cos 2\omega\tau)^2\\ &+ (b\tau\sin(\omega\tau) + c^2\tau\sin(2\omega\tau))^2,\end{aligned} \tag{26}$$

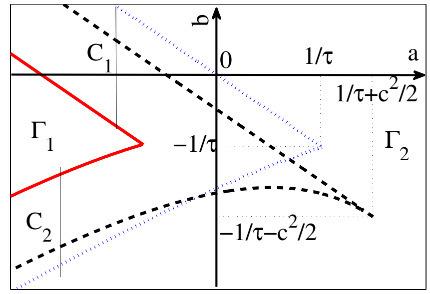

Fig. 2. D-Subdivision for (20) with different c. Two regions Γ_1 and Γ_2 show the stable and unstable parameters region for the system (10) respectively. C_1 and C_2 are the boundaries of stable for $\omega = 0$ and $\omega \in (0,\pi/\tau)$.

we obtain

$$\left(\frac{\text{d}}{\text{d}b}\text{Re}(\lambda)\right)_{\lambda=j\omega} = \frac{1}{\Phi(\omega)}(b\tau + (1 + c^2\tau)\cos(\omega\tau)). \tag{27}$$

Since $\Phi(\omega) > 0$, we just need to consider the sign of $f(\omega) := b\tau + (1 + c^2\tau)\cos(\omega\tau)$ to find the invariance of the sign of $\text{Re}(\lambda)$.

At first, for curve C_1 in Fig. 2, it is easy to see

$$f(\omega)|_{\omega=0} = b\tau + (1 + c^2\tau) = \frac{c^2\tau}{2} > 0 \tag{28}$$

with $b > -1/\tau - c^2/2$. So, by the D-subdivision approach, the number of roots of equation (20) with positive real part will increase as the boundaries C_1 are crossed from down to the up side. That is to say, below C_1 is the stable region.

Then, for curve C_2 in Fig. 2, i.e., $\omega \in (0,\pi/\tau)$, by (23), we have

$$a < \frac{1}{\tau} + \frac{c^2}{2} \quad \text{or} \quad \cos(\omega\tau) < \frac{\sin(\omega\tau)}{\omega\tau}. \tag{29}$$

So,

$$\begin{aligned} f(\omega) =& b\tau + \cos(\omega\tau) + c^2\tau\cos(\omega\tau)\\ =& \left(-c^2\cos(\omega\tau) - \frac{\omega}{\sin(\omega\tau)}\right)\tau + \cos(\omega\tau)\\ &+ c^2\tau\cos(\omega\tau)\\ <& -\frac{\omega\tau}{\sin(\omega\tau)} + \frac{\sin(\omega\tau)}{\omega\tau}\\ <& 0. \end{aligned} \tag{30}$$

The number of roots of (20) with positive real part will decrease as the boundaries C_2 are crossed from down to the up side.

Substituting (23) into (22), for $c \neq 0$, we have

$$\left(c^2 - \frac{\omega}{\sin(\omega\tau)}\right)(1 - \cos(\omega\tau)) = 0. \tag{31}$$

An asymptotic intersection is found as $(c^2/2 + 1/\tau, -c^2/2 - 1/\tau)$ for $\omega = 0$. We can say the second crossing point will not appears until $c > 1/\sqrt{\tau}$. So, the asymptotic

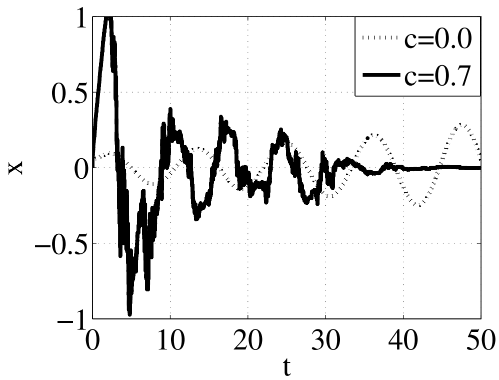

Fig. 3. Solutions of (10) with $a = 0.3, b = -0.65, c = 0.0$ and $c = 0.7$, respectively.

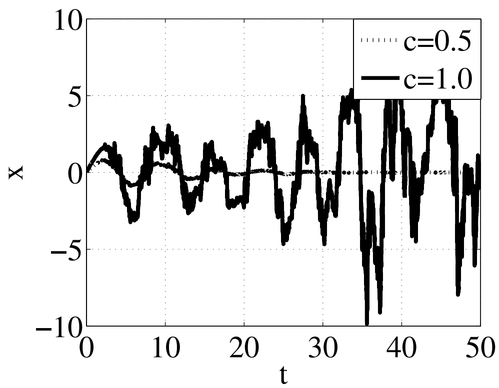

Fig. 4. Solutions of (10) with $a = -0.3, b = -0.65, c = 0.5$ and $c = 1.0$, respectively.

intersection is the only one crossing point of the two curves for $c \in [0, 1/\sqrt{\tau}]$.

For $\omega \neq 0$, if $c > 0$ is big enough then the crossing point could be found for a given ω. By (31), considering the crossing point, there is $\sin(\omega\tau) = \omega/c^2$, because of $d\sin(\omega\tau) = \tau d\omega$ for $\omega = 0$, then the second crossing point appears if and only if

$$c^2 > 1/\tau \text{ or } c > 1/\sqrt{\tau}. \tag{32}$$

By equation (31), for the two curves in the plant a, b have two crossing points for $c > 1/\sqrt{\tau}$. One of them is got for $\omega = 0$, and the other one is

$$\begin{cases} a = \dfrac{c^2}{2} + \omega_0 \dfrac{\cos(\omega_0\tau)}{\sin(\omega_0\tau)} \\ b = -c^2\cos(\omega_0\tau) - \dfrac{\omega_0}{\sin(\omega_0\tau)}, \end{cases} \tag{33}$$

where ω_0 is the single solution of $c^2 \sin(\omega_0\tau) = \omega_0$.

The stable condition (13) and (14) are derived from boundaries C_1 and C_2 then the proof is complete. ■

IV. NUMERICAL EXAMPLES

To illustrate our result developed in Section III, we consider the several numerical examples in this section. Apply the Euler-Maruyama method [10] to the stochastic delay differential equation (10), we obtain

$$x_j = x_{j-1} + (ax_{j-1} + bx_{j-m})\Delta t + cx_{j-m}\Delta w_j, \tag{34}$$

where $\Delta t = T/N, m = \tau/\Delta t$, and T, N denote sample time length and steps, respectively, $\Delta w_j = w_j - w_{j-1}$ which is generated from a discretized brownian path.

From the inequality (32), a critical value $1/\sqrt{\tau}$ can be found for c. The stable region can not be enlarged if $c > 1/\sqrt{\tau}$. Here, we take $\tau = 2$ for the time delay and then $c < 0.707$ can stabilize the stochastic delay system (10). An example illustrated in Fig. 3 shows this result. But if $c > 0$ is bigger than this critical, the stochastic item of the system still play an bad role in the stability analysis, see Fig. 4.

V. CONCLUSION

For a kind of stochastic delay differential equation (10), applying the Ito's formula and D-subdivision approach, we compute the stable region of parametric in the sense of mean square stability. The main results were given in Theorem 1 of Section III. An interesting result was shown by the result that appropriate power of noise will make some unstable areas become to stable in the parametric plant as shown in Fig. 2. That is to say, the time delay and noise are usually exist and could never been cleared, but we don't need to get rid of all kinds of noise. The exist of white noise with appropriate power can stabilize the stochastic delay system. But we have to say that the noise can not stabilize a stochastic differential equation without time delay under the definition of mean square.

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REFERENCES

[1] M. C. Mackey and I. G. Nechaeva, Noise and stability in differential delay equations, *Journal of Dynamics and Differential Equations*, Vol. 6, No. 3, (1994).

[2] M. C. Mackey and I. G. Nechaeva, Solution moment stability in stochastic differential delay equations, *Physical Review E*, Vol. 52, No. 4, (1995).

[3] A. F. Ivanov and A. V. Swishchuk, On global stability of a stochastic differential delay equation, *Banff International Research Station for Mathematical Innovation and Discovery*, June 21-June 28, (2009).

[4] G. Stepan, Delay effects in the human sensory system during balancing, *Philosophical Transactions of the Royal Society*, A367, pp. 1195-1212, (2009).

[5] S. Ushida, J. Lee and K. Deguchi, Performance improvements of stochastic control systems with a dynamics fluctuations, the sixth conference of SICE Journal of Control, Measurement, and System Integration, May,31-June, 2, (2006).

[6] X. R. Mao, C. Selfridge, Stability of stochastic interval systems with time delays. *System & Control Letters* 42, pp. 279-290, (2001).

[7] L. J. Hu and X. R. Mao, Almost sure exponential stabilisation of stochastic systems by statefeedback control. *Automatica*, 44 (2). pp. 465-471, ISSN 0005-1098, (2008).

[8] B. Oksendal, Stochastic differential equations: an introduction with applications, sixth edition, Springer, 6th ed. ,2003. Corr. 3rd, (2005).

[9] M. Bambi, Endogenous growth and time to build: the AK case. *Journal of Economic Dynamics and Control*, Vol. 32, No.4, (2008).

[10] J. H. Desmond, An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations, *SIAM Review*, Vol. 43, No.3, pp. 525-546, (2001).