Littlewood–Paley functions on homogeneous groups

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LITTLEWOOD-PALEY FUNCTIONS ON HOMOGENEOUS GROUPS

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Abstract. We prove L^p estimates for a class of Littlewood-Paley functions on homogeneous groups under a sharp integrability condition of the kernel. The results obtained in the present paper essentially improve some known results.

1. Introduction

We consider the Littlewood-Paley function on $\mathbb{R}^n$ defined by

$$S_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$ and ψ is a function in $L^1(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} \psi(x)\, dx = 0. \tag{1.1}$$

Let

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}$$

be the Poisson kernel for the upper half space $\mathbb{R}^n \times (0, \infty)$. Define $S_\psi(f)$ with

$$\psi(x) = \left(\frac{\partial}{\partial t} P_t(x)\right)_{t=1}.$$

Then S_ψ is a version of the Littlewood-Paley g function. Another classical Littlewood-Paley function is the Marcinkiewicz integral

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3}\right)^{1/2}, \quad F(x) = \int_0^x f(y)\, dy,$$

which can be realized as $S_\psi(f)$ by choosing ψ to be the Haar function on $\mathbb{R}$:

$$\psi(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x),$$

where χ_E denotes the characteristic function of a set E. We refer to [27, 23, 24] for background materials.

Let us recall a result of Benedek, Calderón and Panzone [2] on sufficient conditions for L^p boundedness of S_ψ.

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Theorem A. *In addition to the cancellation condition* (1.1)*, if we assume that*

$$|\psi(x)| \le C(1+|x|)^{-n-\epsilon} \qquad \text{for some} \quad \epsilon > 0, \tag{1.2}$$

$$\int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)|\, dx \le C|y|^{\epsilon} \qquad \text{for some} \quad \epsilon > 0, \tag{1.3}$$

then the operator S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$.

We can easily see that the two classical examples above fulfill the conditions (1.2) and (1.3).

Also, we recall the following result.

Theorem B. *Suppose that ψ satisfies* (1.1) *and* (1.2) *with $\epsilon = 1$. Then S_ψ is bounded on $L^2(\mathbb{R}^n)$.*

This can be found in Coifman and Meyer [5]; see [5, p. 148] and also Journé [15, pp. 81-82], where a proof of Theorem B can be found. We note that in Theorem B the condition (1.3) concerning the regularity of ψ is not assumed.

Theorems A and B are improved by [17] as follows.

Theorem C. *Suppose that ψ satisfies* (1.1) *and* (1.2)*. Then S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$; furthermore, S_ψ is bounded on $L^p_w(\mathbb{R}^n)$ for any $p \in (1,\infty)$ and any $w \in A_p$ (the weight class of Muckenhoupt), where $L^p_w(\mathbb{R}^n)$ denotes the weighted L^p space of all functions f such that $\|f\|_{L^p_w} = \|fw^{1/p}\|_p < \infty$.*

For the rest of this note we assume that ψ is compactly supported. The class $L(\log L)^\alpha(\mathbb{R}^n)$, $\alpha > 0$, is defined to be the collection of all functions f on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |f(x)| [\log(2 + |f(x)|)]^\alpha \, dx < \infty.$$

Similarly, let $L(\log L)^\alpha(S^{n-1})$ be the class of all functions Ω on S^{n-1} satisfying

$$\int_{S^{n-1}} |\Omega(\theta)| \, [\log(2 + |\Omega(\theta)|)]^\alpha \, d\sigma(\theta) < \infty,$$

where $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere of $\mathbb{R}^n$ and $d\sigma$ denotes the Lebesgue surface measure on S^{n-1}.

The following result was proved in [18].

Theorem D. *If ψ is in $L(\log L)^{1/2}(\mathbb{R}^n)$ and satisfies* (1.1)*, then S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $2 \le p < \infty$.*

When $\psi \in L^q(\mathbb{R}^n)$ for some $q > 1$, L^p boundedness of S_ψ for $p \in [2,\infty)$ was proved in [12]. Theorem D is an improvement. On the other hand, for $p < 2$, a result of Duoandikoetxea [8] is known.

Theorem E. *Let r' be the exponent conjugate to r, $1 \le r \le \infty$. Then, we have the following.*

1. *Let $1 < q \le 2$ and $0 < 1/p < 1/2 + 1/q'$. If ψ is in $L^q(\mathbb{R}^n)$ and satisfies* (1.1)*, then S_ψ is bounded on $L^p(\mathbb{R}^n)$.*
2. *If $1 < q < 2$ and $1/p > 1/2 + 1/q'$, then we can find $\psi \in L^q(\mathbb{R}^n)$ such that S_ψ is not bounded on $L^p(\mathbb{R}^n)$.*

Part (1) of Theorem E improves a previous result in [3] and was proved by applying a weight theory (see also [10]). Define $\psi^{(\alpha)}$ on $\mathbb{R}$ by

$$\psi^{(\alpha)}(x) = \begin{cases} \alpha(1-|x|)^{\alpha-1}\operatorname{sgn}(x), & x \in (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

If $1 < p < 2$, $1 < q < 2$ and $1/q' < \alpha < 1/p - 1/2$, then $\psi^{(\alpha)} \in L^q(\mathbb{R})$ and $S_{\psi^{(\alpha)}}$ is not bounded on L^p; this follows from Remark 2 of [12].

Let

$$\psi(x) = |x|^{-n+1}\Omega(x')\chi_{(0,1]}(|x|) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\},$$

where $x' = x/|x|$, $\Omega \in L^1(S^{n-1})$, $\int_{S^{n-1}} \Omega\, d\sigma = 0$. Then, the Littlewood-Paley function $S_\psi(f)$ is the Marcinkiewicz integral $\mu_\Omega(f)$ in Stein [22] (see also Hörmander [14]).

For μ_Ω we recall a result of Al-Salman, Al-Qassem, Cheng and Pan [1].

Theorem F. *If $\Omega \in L(\log L)^{1/2}(S^{n-1})$, then μ_Ω is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1,\infty)$.*

The case $p = 2$ of Theorem F is due to Walsh [26].

We can also consider Littlewood-Paley functions on homogeneous groups. Let $n \geq 2$. We also regard $\mathbb{R}^n$ as a homogeneous group. Multiplication of the Lie group is given by a polynomial mapping and the underlying manifold is $\mathbb{R}^n$ itself. We denote by $\mathbb{H}$ the homogeneous group. We recall that $\mathbb{H}$ admits a dilation family $\{A_t\}_{t>0}$ of the form

$$A_t x = (t^{a_1}x_1, t^{a_2}x_2, \dots, t^{a_n}x_n), \quad x = (x_1, \dots, x_n),$$

where $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, such that each A_t is an automorphism of the group structure, which requires

$$A_t(xy) = (A_t x)(A_t y), \quad x, y \in \mathbb{H}, t > 0$$

(see [13, 16]). $\mathbb{H}$ has a homogeneous nilpotent Lie group structure, where Lebesgue measure is a bi-invariant Haar measure, the identity is the origin 0 and $x^{-1} = -x$.

We can define a norm function r on $\mathbb{H}$ satisfying the following conditions:

(i) $r(x) \geq 0$ for all $x \in \mathbb{H}$, $r(x) = 0$ if and only if $x = 0$;
(ii) r is continuous on $\mathbb{H}$ and is C^∞ in $\mathbb{H} \setminus \{0\}$;
(iii) $r(A_t x) = t r(x)$ for all $t > 0$ and $x \in \mathbb{H}$;
(iv) $r(x) = r(x^{-1})$ for all $x \in \mathbb{H}$.

Moreover, we may assume that $\Sigma = \{x \in \mathbb{H} : r(x) = 1\}$ coincides with S^{n-1}. Let $\gamma = a_1 + \cdots + a_n$ (the homogeneous dimension of $\mathbb{H}$). Then, we have the formula

$$\int_{\mathbb{H}} f(x)\,dx = \int_0^\infty \int_\Sigma f(A_t\theta)t^{\gamma-1}\,dS(\theta)\,dt, \quad dS = \omega\,d\sigma, \tag{1.4}$$

where ω is a strictly positive C^∞ function on S^{n-1} and $d\sigma$ is the Lebesgue surface measure on S^{n-1} as above. The convolution is defined by

$$f * g(x) = \int_{\mathbb{H}} f(y)g(y^{-1}x)\,dy.$$

We refer to [6, 20, 25] for more details.

For a function f on $\mathbb{H}$, let

$$f_t(x) = \delta_t f(x) = t^{-\gamma} f(A_t^{-1}x).$$

We consider the Littlewood-Paley function on $\mathbb{H}$ defined by

$$S_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where ψ is in $L^1(\mathbb{H})$ and satisfies (1.1). Let Ω be locally integrable in $\mathbb{H} \setminus \{0\}$. We assume that Ω is homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, which means that $\Omega(A_t x) = \Omega(x)$ for $x \neq 0$, $t > 0$. Also, we require that

$$\int_\Sigma \Omega(\theta)\, dS(\theta) = 0. \tag{1.5}$$

The space $L(\log L)^\alpha(\Sigma)$ can be defined as above with respect to the measure dS.

If Ψ is defined by

$$\Psi(x) = r(x)^{-\gamma+a}\Omega(x')\chi_{(0,1]}(r(x)), \quad a > 0, \tag{1.6}$$

then we also write $\mu_\Omega = S_\Psi$, where $x' = A_{r(x)^{-1}}x$ for $x \neq 0$.

Ding-Wu [7] proved the following.

Theorem G. *Let Ω be in $L\log L(\Sigma)$ and satisfy* (1.5). *Define μ_Ω by Ψ in* (1.6) *with $a = 1$. Then*

(1) *μ_Ω is bounded on $L^p(\mathbb{H})$ for $p \in (1, 2]$;*
(2) *μ_Ω is of weak type* $(1, 1)$.

In this note we shall prove the following.

Theorem 1. *Suppose that Ω is in $L(\log L)^{1/2}(\Sigma)$ and satisfies* (1.5). *Then μ_Ω is bounded on $L^p(\mathbb{H})$ for all $p \in (1,\infty)$.*

Obviously, Theorem 1 improves part (1) of Theorem G essentially since $L\log L(\Sigma)$ is a proper subspace of $L(\log L)^{1/2}(\Sigma)$.

Following [11], to prove Theorem 1 we decompose $\Psi(x) = \sum_{k<0} 2^{ka}\Psi^{(k)}(x)$, $k \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers and

$$\Psi^{(k)}(x) = 2^{-ka}r(x)^{a-\gamma}\Omega(x')\chi_{(1,2]}(2^{-k}r(x)).$$

Note that for any $k \in \mathbb{Z}$,

$$S_{\Psi^{(k)}}(f)(x) = S_{\Psi^{(k)}_{2^{-k}}}(f)(x) = S_{\Psi^{(0)}}(f)(x),$$

and hence

$$S_\Psi(f)(x) \leq \sum_{k<0} 2^{ka} S_{\Psi^{(k)}}(f)(x) = c_a S_{\Psi^{(0)}}(f)(x).$$

This observation suggests to consider a function of the form

$$\Psi(x) = \ell(r(x))\frac{\Omega(x')}{r(x)^\gamma}, \tag{1.7}$$

where ℓ is in Λ^η_∞ (see [20]) for some $\eta > 0$ and supported in the interval $[1, 2]$.

Here we recall the definition of Λ^η_q from [20]. Let d_s, $1 \leq s \leq \infty$, denote the collection of all measurable functions h on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ satisfying $\|h\|_{d_s} < \infty$, where

$$\|h\|_{d_s} = \sup_{j\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s \frac{dt}{t}\right)^{1/s}$$

if $1 \le s < \infty$, and $\|h\|_{d_\infty} = \|h\|_{L^\infty(\mathbb{R}_+)}$. Then $d_s \subset d_u$ if $s \ge u$. For $t \in (0,1]$, define

$$\omega(h,t) = \sup_{|s|<tR/2} \int_R^{2R} |h(r-s) - h(r)| \frac{dr}{r},$$

where the supremum is taken over all s and R such that $|s| < tR/2$ (see [21]). Let $\eta > 0$ and define Λ^η to be the family of all locally integrable functions h on $\mathbb{R}_+$ satisfying

$$\|h\|_{\Lambda^\eta} = \sup_{t\in(0,1]} t^{-\eta}\omega(h,t) < \infty.$$

Let $\Lambda_s^\eta = d_s \cap \Lambda^\eta$ with $\|h\|_{\Lambda_s^\eta} = \|h\|_{d_s} + \|h\|_{\Lambda^\eta}$ for $h \in \Lambda_s^\eta$. Then $\Lambda_s^{\eta_1} \subset \Lambda_s^{\eta_2}$ if $\eta_2 \le \eta_1$, and $\Lambda_{s_1}^\eta \subset \Lambda_{s_2}^\eta$ if $s_2 \le s_1$.

To prove Theorem 1 it suffices to show the following.

Theorem 2. *Let Ψ be defined by* (1.7). *We assume that Ω is in $L(\log L)^{1/2}(\Sigma)$ and satisfies* (1.5). *Then S_Ψ is bounded on $L^p(\mathbb{H})$ for all $p \in (1,\infty)$.*

We shall prove Theorem 2 via extrapolation arguments using the following estimates.

Theorem 3. *Let Ψ be as in* (1.7). *We assume that Ω is in $L^s(\Sigma)$ for some $s \in (1,2]$ and satisfies* (1.5). *Let $1 < p < \infty$. Then*

$$\|S_\Psi(f)\|_p \le C_p(s-1)^{-1/2}\|\Omega\|_s\|f\|_p,$$

where the space $L^s(\Sigma)$ is defined with respect to the measure dS and the constant C_p is independent of s and Ω.

Indeed, Theorem 3 implies Theorem 2 as follows. Let Ω and Ψ be as in Theorem 2. We can decompose Ω as

$$\Omega = \sum_{m=1}^\infty b_m\Omega_m,$$

where each Ω_m satisfies (1.5) and $\{b_m\}$ is a sequence of non-negative real numbers such that $\sum_{m=1}^\infty m^{1/2}b_m < \infty$, furthermore $\sup_{m\ge1}\|\Omega_m\|_{1+1/m} \le 1$ (see Lemma 3 of [19]). Accordingly,

$$\Psi = \sum_{m=1}^\infty \Psi_m, \quad \Psi_m(x) = b_m\ell(r(x))\frac{\Omega_m(x')}{r(x)^\gamma}.$$

Let $1 < p < \infty$. By Theorem 3 with $s = 1 + 1/m$ we have

$$\|S_{\Psi_m}(f)\|_p \le C_p m^{1/2} b_m\|\Omega_m\|_{1+1/m}\|f\|_p \le C_p m^{1/2}b_m\|f\|_p,$$

which implies

$$\|S_\Psi(f)\|_p \le \sum_{m=1}^\infty \|S_{\Psi_m}(f)\|_p \le C_p(\sum_{m=1}^\infty m^{1/2}b_m)\|f\|_p.$$

This will complete the proof of Theorem 2.

In Section 2, we shall prove some vector valued inequalities on the homogeneous groups. To prove these inequalities we argue as in Sections 3, 4 of [12] and we apply L^p estimates of M. Christ [4] for the maximal functions along homogeneous curves. We shall prove Theorem 3 in Section 3. The methods for the proof of Theorem 3 have some similarities to those employed in [6, 20] in studying L^p boundedness of singular integrals on homogeneous groups. Outline of the proof is based on

Duoandikoetxea and Rubio de Francia [9], so we apply the vector valued inequalities of Section 2, Littlewood-Paley decompositions and orthogonality arguments for L^2 estimates; however the L^2 estimates will be obtained not through the Fourier transform but convolution. A basic L^2 estimate for the orthogonality (Lemma 9) will be proved by the methods of T. Tao [25]. Finally, in Section 4 we shall show that Theorem 1 can be applied to prove L^p boundedness of some other Littlewood-Paley functions.

2. Vector valued inequalities

Let $p(x) = \int_1^\rho |\psi_t(x)|\, dt/t$, $\rho \geq 2$. Define a maximal function

$$N_\psi^{(\rho)}(f)(x) = \sup_{k\in\mathbb{Z}} |(f * p_{\rho^k})(x)|.$$

Let $\mathcal{H}$ denote the Hilbert space $L^2((0,\infty), dt/t)$. For each $k \in \mathbb{Z}$ we consider an operator T_k defined by

$$(T_k(f)(x))\,(t) = T_k(f)(x,t) = f * \psi_t(x)\chi_{[1,\rho)}(\rho^{-k}t). \tag{2.1}$$

The operator T_k maps functions on $\mathbb{H}$ to $\mathcal{H}$-valued functions on $\mathbb{H}$ and we see that

$$|T_k(f)(x)|_{\mathcal{H}} = \left(\int_{\rho^k}^{\rho^{k+1}} |f * \psi_t(x)|^2 \,\frac{dt}{t}\right)^{1/2} = \left(\int_1^\rho |f * \psi_{\rho^k t}(x)|^2 \,\frac{dt}{t}\right)^{1/2}.$$

Lemma 1. *Let $2 < s < \infty$, $r = (s/2)' = s/(s-2)$. If $N_{\tilde\psi}^{(\rho)}$ is bounded on $L^r(\mathbb{H})$, then*

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2\right)^{1/2} \right\|_s \leq \|\psi\|_1^{1/2} B_{\rho,r}^{1/2}(\tilde\psi) \left\| \left(\sum_k |f_k|^2\right)^{1/2} \right\|_s,$$

where $\tilde\psi(x) = \psi(x^{-1})$ and $B_{\rho,r}(\tilde\psi)$ is the operator norm of $N_{\tilde\psi}^{(\rho)}$ on $L^r(\mathbb{H})$.

Proof. Take a non-negative $g \in L^r$ satisfying $\|g\|_r \leq 1$ and

$$I := \left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2\right)^{1/2} \right\|_s^2 = \int \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2\right) g\, dx.$$

Then since

$$|T_k(f_k)|_{\mathcal{H}}^2 \leq \|\psi\|_1 \int_{\mathbb{H}} |f_k(y)|^2 p_{\rho^k}(y^{-1}x)\, dy,$$

we have

$$\begin{aligned} I &\leq \|\psi\|_1 \sum_k \int |f_k(y)|^2 \left(\int_{\mathbb{H}} p_{\rho^k}(y^{-1}x) g(x)\, dx\right) dy \\ &\leq \|\psi\|_1 \sum_k \int |f_k(y)|^2 N_{\tilde\psi}^{(\rho)}(g)(y)\, dy, \end{aligned} \tag{2.2}$$

where we note that

$$N_{\tilde\psi}^{(\rho)}(g)(y) = \sup_{k\in\mathbb{Z}} \left| \int_{\mathbb{H}} g(x) p_{\rho^k}(y^{-1}x)\, dx \right|.$$

Hölder's inequality implies

$$\sum_k \int |f_k(y)|^2 N_{\tilde\psi}^{(\rho)}(g)(y)\,dy \le \left\| \left(\sum_k |f_k|^2\right)^{1/2} \right\|_s^2 \|N_{\tilde\psi}^{(\rho)}(g)\|_r \le B_{\rho,r}(\tilde\psi) \left\| \left(\sum_k |f_k|^2\right)^{1/2} \right\|_s^2 . \tag{2.3}$$

Combining (2.2) and (2.3), we get the conclusion. □

Define a maximal function

$$M_\psi(f)(x) = \sup_{t>0} |f * |\psi|_t(x)| .$$

Lemma 2. *Let $1 < s < 2$, $r = (s'/2)' = s/(2-s)$. Suppose that M_ψ is bounded on $L^r(\mathbb{H})$. Then*

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2\right)^{1/2} \right\|_s \le (\log\rho)^{1/2} \|\psi\|_1^{1/2} C_r(\psi)^{1/2} \left\| \left(\sum_k |f_k|^2\right)^{1/2} \right\|_s ,$$

where $C_r(\psi)$ is the operator norm of M_ψ on $L^r(\mathbb{H})$.

If h is a function on $\mathbb{H}\times(0,\infty)$, define an $\mathcal{H}$-valued function $P_k(h)$ by

$$(P_k(h)(x))(t) = P_k(h)(x,t) = h(x,t)\chi_{[1,\rho)}(\rho^{-k}t).$$

Also, we define $T_k(h)$ by $(T_k(h)(x))(t) = T_k(h)(x,t) = T_k(h(\cdot,t))(x)$.

Proof of Lemma 2 requires the following.

Lemma 3. *For a sequence $\{h_k(x,t)\}$ of functions on $\mathbb{H}\times(0,\infty)$ we have*

$$\left\| \left(\sum_k |\tilde T_k(h_k)|_{\mathcal{H}}^2\right)^{1/2} \right\|_{s'} \le \|\psi\|_1^{1/2} C_r(\psi)^{1/2} \left\| \left(\sum_k |P_k(h_k)|_{\mathcal{H}}^2\right)^{1/2} \right\|_{s'}$$

under the assumptions of Lemma 2, where $\tilde T_k h$ is defined as $T_k h$ with $\bar\psi(y^{-1})$ in place of $\psi(y)$.

Proof. Choose a non-negative $g \in L^r$ satisfying $\|g\|_r \le 1$ and

$$\left\| \left(\sum_k |\tilde T_k(h_k)|_{\mathcal{H}}^2\right)^{1/2} \right\|_{s'}^2 = \int \left(\sum_k |\tilde T_k(h_k)(x)|_{\mathcal{H}}^2\right) g(x)\,dx. \tag{2.4}$$

We observe that

$$\int |\tilde T_k(h_k)(x)|_{\mathcal{H}}^2 g(x)\,dx \le \|\psi\|_1 \int M_\psi(g)(x) |P_k(h_k)(x)|_{\mathcal{H}}^2\,dx.$$

Thus, applying Hölder's inequality, we have

$$\int \left(\sum_k |\tilde{T}_k(h_k)(x)|^2_{\mathcal{H}}\right) g(x)\,dx \le \|\psi\|_1 \left\| \left(\sum_k |P_k(h_k)|^2_{\mathcal{H}}\right)^{1/2} \right\|^2_{s'} \|M_\psi(g)\|_r \tag{2.5}$$
$$\le \|\psi\|_1 C_r(\psi) \left\| \left(\sum_k |P_k(h_k)|^2_{\mathcal{H}}\right)^{1/2} \right\|^2_{s'}.$$

The inequality claimed follows from (2.4) and (2.5). □

We can give the proof of Lemma 2 now. Let $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denote the inner product in $\mathcal{H}$. We observe that

$$\int \langle T_k(f_k)(x,\cdot), h_k(x,\cdot)\rangle_{\mathcal{H}}\,dx = \int \langle P_k(f_k)(x,\cdot), \tilde{T}_k(h_k)(x,\cdot)\rangle_{\mathcal{H}}\,dx,$$

where $P_k(f_k)(x,t) = f_k(x)\chi_{[1,\rho)}(\rho^{-k}t)$. Note that

$$\left(\sum_k |P_k(f_k)|^2_{\mathcal{H}}\right)^{1/2} = (\log\rho)^{1/2}\left(\sum_k |f_k|^2\right)^{1/2}.$$

Thus, Hölder's inequality and Lemma 3 imply that

$$\left|\int \sum_k \langle T_k(f_k)(x,\cdot), h_k(x,\cdot)\rangle_{\mathcal{H}}\,dx\right|$$
$$\le (\log\rho)^{1/2}\|\psi\|_1^{1/2} C_r(\psi)^{1/2} \left\|\left(\sum_k |f_k|^2\right)^{1/2}\right\|_s \left\|\left(\sum_k |P_k(h_k)|^2_{\mathcal{H}}\right)^{1/2}\right\|_{s'}.$$

The conclusion of Lemma 2 follows from this by taking the supremum over $\{h_k(x,t)\}$ with $\left\|\left(\sum_k |P_k(h_k)|^2_{\mathcal{H}}\right)^{1/2}\right\|_{s'} \le 1$.

Let Ψ be as in (1.7). We shall use the following estimates of M_Ψ.

Lemma 4. *Suppose that Ω is in $L^1(\Sigma)$ (the condition (1.5) is not needed). Then*

$$\|M_\Psi f\|_p \le C_p\|\Omega\|_1\|f\|_p$$

for $p > 1$.

For $\theta \in \Sigma$, we define

$$M_\theta f(x) = \sup_{s>0} \frac{1}{s}\int_0^s |f(x(A_t\theta)^{-1})|\,dt.$$

To prove Lemma 4 we need the following result of [4].

Lemma 5. *There exists a constant C_p independent of θ such that*

$$\|M_\theta f\|_p \le C_p\|f\|_p$$

for $p > 1$.

Proof of Lemma 4. By a change of variables (see (1.4)), we have

$$f * |\Psi|_t(x) = \int f(xy^{-1})|\Psi|_t(y)\,dy = \int_1^2\int_\Sigma f(x(A_{st}\theta)^{-1})|\Omega(\theta)\ell(s)|s^{-1}\,dS(\theta)\,ds.$$

It is easy to see that

$$\int_1^2 |f(x(A_{st}\theta)^{-1})||\ell(s)|s^{-1}\,ds \le \|\ell\|_\infty \frac{1}{t}\int_t^{2t} |f(x(A_s\theta)^{-1})|\,ds.$$

Thus

$$M_\Psi f(x) \le C\|\ell\|_\infty \int_\Sigma M_\theta f(x)|\Omega(\theta)|\,dS(\theta).$$

This estimate and Minkowski's inequality imply the conclusion. □

Note that

$$N_\Psi^{(\rho)} f(x) \le C(\log\rho) M_\Psi f(x).$$

Thus by Lemmas 1, 2 and 4 we have the following.

Lemma 6. *If* $1 < s < \infty$, *then*

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_s \le C(\log\rho)^{1/2}\|\Omega\|_1 \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s,$$

where each T_k *is defined as in* (2.1) *with* Ψ *of* (1.7) *in place of* ψ.

3. Proof of Theorem 3

Let ϕ be a C^∞ function supported in $\{1/2 < r(x) < 1\}$. We assume that $\int \phi = 1$, $\phi(x) = \tilde{\phi}(x)$, $\phi(x) \ge 0$ for all $x \in \mathbb{H}$. For $\rho \ge 2$, we define

$$\Delta_k = \delta_{\rho^{k-1}}\phi - \delta_{\rho^k}\phi, \quad k \in \mathbb{Z}.$$

Then Δ_k is supported in $\{\rho^{k-1}/2 < r(x) < \rho^k\}$ and $\Delta_k = \tilde{\Delta}_k$. Also, we easily see that $\sum_k \Delta_k = \delta$, where δ is the delta function.

Let Ψ be as in (1.7) and satisfy the assumptions of Theorem 3. We decompose

$$f * \Psi_t(x) = \sum_{j\in\mathbb{Z}} F_j(x,t),$$

where

$$F_j(x,t) = \sum_{k\in\mathbb{Z}} f * \Delta_{j+k} * \Psi_t(x)\chi_{[\rho^k,\rho^{k+1})}(t).$$

Define

$$U_j f(x) = \left(\int_0^\infty |F_j(x,t)|^2 \frac{dt}{t} \right)^{1/2} = \left(\sum_{k\in\mathbb{Z}} \int_1^\rho |f * \Delta_{j+k} * \Psi_{\rho^k t}|^2 \frac{dt}{t} \right)^{1/2}$$

$$= \left(\sum_k |T_k(f * \Delta_{j+k})|_{\mathcal{H}}^2 \right)^{1/2},$$

where T_k is as in (2.1) with ψ_t replaced by Ψ_t.

To prove Theorem 3 we apply the following estimates.

Lemma 7. *Let* $1 < s \le 2$ *and* $\rho = 2^{s'}$. *Then we have*

$$\|U_j f\|_2 \le C(s-1)^{-1/2} 2^{-\epsilon|j|}\|\Omega\|_s\|f\|_2,$$

where ϵ, C *are positive constants independent of* s, $\Omega \in L^s(\Sigma)$ *and* $j \in \mathbb{Z}$.

Let $\psi_j \in C_0^\infty(\mathbb{R})$, $j \in \mathbb{Z}$, be such that

$$\psi_j \geq 0,$$
$$\operatorname{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j < t < \rho^{j+2}\},$$
$$\log 2 \sum_{j\in\mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t > 0,$$
$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots,$$

where c_m is a constant independent of ρ (this is possible since $\rho \geq 2$). We may assume that $\psi_j(t) = \psi_0(\rho^{-j}t)$.

Let

$$K_0(x) = \frac{\Omega(x')}{r(x)^\gamma} \chi_{[1,2]}(r(x))$$

and decompose

$$\frac{\Omega(x')}{r(x)^\gamma} = \sum_{j\in\mathbb{Z}} S_j(x), \quad S_j(x) = \int_0^\infty \psi_j(t)\delta_t K_0(x)\,\frac{dt}{t} = \frac{\Omega(x')}{r(x)^\gamma} \int_{1/2}^1 \psi_j(tr(x))\,\frac{dt}{t}.$$

We note that S_j is supported in $\{\rho^j < r(x) < 2\rho^{j+2}\}$. Let

$$L_m^{(t)}(x) = \ell(t^{-1}r(x)) S_m(x).$$

Then by the support condition we have

$$\Psi_t(x)\chi_{[\rho^k,\rho^{k+1})}(t) = \sum_{m=k-2}^{k+1} L_m^{(t)}(x)\chi_{[\rho^k,\rho^{k+1})}(t)$$

and

$$F_j(x,t) = \sum_{k\in\mathbb{Z}} \sum_{m=k-2}^{k+1} f * \Delta_{j+k} * L_m^{(t)}(x)\chi_{[\rho^k,\rho^{k+1})}(t).$$

Lemma 7 will be derived from the following.

Lemma 8. *Let $1 < s \leq 2$ and $-2 \leq m \leq 1$. Let $\{\sigma_k\}$ be a sequence of real numbers such that $\sigma_k = 1$ or -1. Define an operator $R_j^{(t)}$ by*

$$R_j^{(t)} f(x) = \sum_{k\in\mathbb{Z}} \sigma_k f * \Delta_{j+k} * L_{k+m}^{(\rho^k t)}(x).$$

Then, if $\rho = 2^{s'}$,

$$\|R_j^{(t)} f\|_2 \leq C 2^{-\epsilon|j|} \|\Omega\|_s \|f\|_2,$$

where the constants ϵ, C are positive and independent of s, $\Omega \in L^s(\Sigma)$, $j \in \mathbb{Z}$, $t \in [1,\rho)$ and the sequence $\{\sigma_k\}$.

To prove Lemma 8 we use the following estimates.

Lemma 9. *Let $1 < s \leq 2$, $\rho = 2^{s'}$ and $-2 \leq m \leq 1$. Then we have*

$$\|f * L_{k+m}^{(\rho^k t)} * \Delta_{j+k}\|_2 \leq C 2^{-\epsilon|j|} \|\Omega\|_s \|f\|_2$$

for some positive constants ϵ, C independent of s, $\Omega \in L^s(\Sigma)$, $j, k \in \mathbb{Z}$ and $t \in [1,\rho)$.

We now prove Lemma 8 taking Lemma 9 for granted. We may assume that all the functions in question are real-valued. Arguing as in the proof of Lemma 7 of [20] (see p. 321 of [20]) and using Lemma 9 with duality we can show that

$$\left\| f * \Delta_{j+k} * L_{k+m}^{(\rho^k t)} * \Delta_{j'+k} * \Delta_{j'+k'} * \tilde{L}_{k'+m}^{(\rho^{k'} t)} * \Delta_{j+k'} \right\|_2 \le C\|\Omega\|_s^2 2^{-\epsilon|j|} 2^{-\epsilon|j'|} \min(1, \rho^{-\epsilon(|k-k'|-c)/2})\|f\|_2$$

for some $\epsilon > 0$. We have a similar estimate for

$$\left\| f * \Delta_{j'+k'} * \tilde{L}_{k'+m}^{(\rho^{k'} t)} * \Delta_{j+k'} * \Delta_{j+k} * L_{k+m}^{(\rho^k t)} * \Delta_{j'+k} \right\|_2.$$

Thus, the Cotlar-Knapp-Stein lemma implies

$$\|G_{j,j'} f\|_2 \le C\|\Omega\|_s 2^{-\epsilon|j|/2} 2^{-\epsilon|j'|/2}\|f\|_2,$$

where

$$G_{j,j'} f = \sum_{k\in\mathbb{Z}} \sigma_k f * \Delta_{j+k} * L_{k+m}^{(\rho^k t)} * \Delta_{j'+k}$$

and hence

$$\|R_j^{(t)} f\|_2 \le \sum_{j'\in\mathbb{Z}} \|G_{j,j'} f\|_2 \le C(1-2^{-\epsilon/2})^{-1}\|\Omega\|_s 2^{-\epsilon|j|/2}\|f\|_2,$$

which is the assertion of Lemma 8.

Assuming Lemma 8, we can prove Lemma 7 as follows. By Lemma 8 with the random choice of $\{\sigma_k\}$, the Khintchine inequality implies that

$$\left\| \left(\sum_{k\in\mathbb{Z}} |f * \Delta_{j+k} * L_{k+m}^{(\rho^k t)}|^2 \right)^{1/2} \right\|_2^2 \le C 2^{-2\epsilon|j|}\|\Omega\|_s^2 \|f\|_2^2.$$

This estimate is uniform in $t \in [1, \rho)$. Thus, integration over $[1, \rho)$ with respect to the measure dt/t gives

$$\|U_j f\|_2^2 = \int_1^\rho \left\| \left(\sum_{k\in\mathbb{Z}} |f * \Delta_{j+k} * \sum_{m=-2}^{1} L_{k+m}^{(\rho^k t)}|^2 \right)^{1/2} \right\|_2^2 \frac{dt}{t} \le C(\log\rho) 2^{-2\epsilon|j|}\|\Omega\|_s^2 \|f\|_2^2,$$

which proves Lemma 7.

To complete the proof of Lemma 7, it thus remains to prove Lemma 9.

Proof of Lemma 9. For $u \in [1, \rho)$ and $-2 \le m \le 1$, let

$$S^{(u)} = \delta_{\rho^{-k-m}} L_{k+m}^{(\rho^k u)} = \ell(\rho^m u^{-1} r(x)) \delta_{\rho^{-k-m}} S_{k+m} = \ell(\rho^m u^{-1} r(x)) S_0.$$

By the proof of Lemma 2 of [20], we have

$$\|S^{(u)}\|_q \le C(\log\rho)^{1/q'} \|\ell\|_{d_q} \|K_0\|_q, \quad q \ge 1, \tag{3.1}$$

where the constant C is independent of ρ and $u \in [1, \rho)$. This can be easily seen since $\operatorname{supp}(\ell) \subset [1, 2]$. Using (3.1) with $q = 1$ and arguing as in [20], for $j \ge 0$, we see that

$$\|S^{(u)} * \Delta_j\|_1 \le C \min\left(1, \rho^{-\epsilon j + \tau}\right) \|\ell\|_{d_1} \|K_0\|_1 \tag{3.2}$$

for some $\epsilon, \tau > 0$.

Also, for $j < 0$ we have

$$(3.3)\quad \left| \iiint \Delta_k(x) G_1(y,t) g(H(y,t)x) \prod_{i=1}^{n} \left(\psi_0(t_i)\ell(t_i,y_i)K_0(y_i)\right) \, dy\, \bar{d}t\, dx \right| \le C(\log\rho)^{n/s'} \rho^{\epsilon(j+c)/s'} \|\ell\|^n_{\Lambda^\eta_s} \|K_0\|^n_s,$$

$$(3.4)\quad \left| \iiint \Delta_k(x) G_2(y,t) g(H(y,t)x) \prod_{i=1}^{n} \left(\psi_0(t_i)\ell(t_i,y_i)K_0(y_i)\right) \, dy\, \bar{d}t\, dx \right| \le C\rho^{\delta\epsilon(j+c)/s'} \|\ell\|^n_{\Lambda^\eta_s} \|K_0\|^n_1.$$

Here we have used notation similar to that in [20] modified appropriately for the present context; $\ell(t_i, y_i) = \ell(u^{-1}t_i\rho^m r(y_i))$, g is a smooth function with compact support such that $\|g\|_\infty \le 1$, $y = (y_1, \dots, y_n) \in \mathbb{H}^n$, $t = (t_1, \dots, t_n)$, $dy = dy_1 \dots dy_n$, $\bar{d}t = (dt_1/t_1)\dots(dt_n/t_n)$,

$$H(y,t) = w_1 A_{t_1} y_1 \dots w_n A_{t_n} y_n$$

with $w_1, \dots, w_n \in \mathbb{H}$ such that $r(w_k) \le C\rho^2$, $k = 1, 2, \dots, n$, $DH(y,t)$ is the $n \times n$ matrix whose ith column vector is $\partial^L_{t_i} H(y,t)$:

$$DH(y,t) = \left(\partial^L_{t_1} H(y,t), \dots, \partial^L_{t_n} H(y,t)\right),$$

and

$$G_1(y,t) = \zeta_1\left(\rho^{-n\epsilon j}\det(DH(y,t))\right),$$
$$G_2(y,t) = \zeta_2\left(\rho^{-n\epsilon j}\det(DH(y,t))\right),$$

where ζ_1, ζ_2 are functions in $C^\infty(\mathbb{R})$ which satisfy $0 \le \zeta_1 \le 1$, $\mathrm{supp}(\zeta_1) \subset [-1, 1]$, $\zeta_1(t) = 1$ for $t \in [-1/2, 1/2]$, $\zeta_2 = 1 - \zeta_1$, and δ, ϵ are small positive numbers.

To prove (3.3), we argue as in the proof of (3.13) of [20] and we only have to note that

$$\int_0^\infty |\ell(t_i, y_i)|^s \frac{dt_i}{t_i} \le C\|\ell\|^s_{d_s},$$

since ℓ is supported in $[1, 2]$.

To prove (3.4) we recall

$$\tilde{\ell}(t_i, y_i) = \int_{s_i < t_i/2} \ell(t_i - s_i, y_i)\varphi_{\rho^{\epsilon j}}(s_i)\, ds_i,$$

where $\varphi_u(s_i) = u^{-1}\varphi(u^{-1}s_i)$, $u > 0$, with $\varphi \in C^\infty(\mathbb{R})$ satisfying $\mathrm{supp}(\varphi) \subset (0, 1/8)$, $\varphi \ge 0$, $\int \varphi(s)\, ds = 1$. Then, we can easily see that

$$\int \psi_0(t_i)|\ell(t_i, y_i)| \frac{dt_i}{t_i} \le C\|\ell\|_{d_1},$$

$$\int \psi_0(t_i)|\tilde{\ell}(t_i, y_i)| \frac{dt_i}{t_i} \le C\|\ell\|_{d_1}, \quad \int_1^{\rho^2} |\tilde{\ell}(t_i, y_i)| \frac{dt_i}{t_i} \le C\|\ell\|_{d_1},$$

$$\int \psi_0(t_i)|\ell(t_i, y_i) - \tilde{\ell}(t_i, y_i)| \frac{dt_i}{t_i} \le C\rho^{\epsilon\eta j},$$

$$\int_1^{\rho^2} |\partial_{t_i}\tilde{\ell}(t_i, y_i)| \frac{dt_i}{t_i} \le C\rho^{-j\epsilon}\|\ell\|_{d_1}.$$

Using these estimates and arguing as in the proof of (3.14) of [20], we can prove (3.4).

Applying (3.1), (3.2), (3.3) and (3.4) as in the proof of Lemma 1 of [20], we can reach the conclusion of Lemma 9. □

We turn to the proof of Theorem 3. Let $1 < p < \infty$. By Lemma 6 and the Littlewood-Paley inequality (see Lemma 6 of [20])

$$\left\| \left(\sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_r \le C_r \|f\|_r, \quad 1 < r < \infty,$$

where C_r is independent of ρ, we have

$$\|U_j(f)\|_r \le C_r (\log \rho)^{1/2} \|\Omega\|_1 \|f\|_r$$

for all $r \in (1, \infty)$, where U_j is as in Lemma 7. Since we also have the L^2-estimates of Lemma 7, if $\rho = 2^{s'}$ $(1 < s \le 2)$, interpolation will give

$$\|U_j f\|_p \le C(s-1)^{-1/2} 2^{-\epsilon |j|} \|\Omega\|_s \|f\|_p$$

with some $\epsilon > 0$, which implies

$$\|S_\Psi(f)\|_p \le \sum_j \|U_j f\|_p \le C_p (s-1)^{-1/2} \|\Omega\|_s \|f\|_p.$$

This completes the proof of Theorem 3.

4. Applications

Finally, we give some applications of Theorem 1. Firstly, we may get the L^p $(2 \le p < \infty)$ boundedness for the Littlewood-Paley operators A_Ω and $\mu^*_{\Omega,\lambda}$ related to the area integral and the Littlewood-Paley g^*_λ function, respectively. They are defined by

$$A_\Omega(f)(x) = \left(\iint_{\Gamma(x)} |f * \Psi_t(y)|^2 \frac{dydt}{t^{\gamma+1}} \right)^{1/2}$$

and

$$\mu^*_{\Omega,\lambda}(f)(x) = \left(\iint_{\mathbb{H} \times \mathbb{R}_+} \left(\frac{t}{t + r(y^{-1}x)} \right)^{\gamma\lambda} |f * \Psi_t(y)|^2 \frac{dydt}{t^{\gamma+1}} \right)^{1/2}, \qquad \lambda > 1,$$

respectively, where $\Gamma(x) = \{(y,t) \in \mathbb{H} \times \mathbb{R}_+ : r(y^{-1}x) < t\}$ and Ψ is as in (1.6).

Theorem 4. *Suppose that Ω is in $L(\log L)^{1/2}(\Sigma)$ and satisfies (1.5). Then $\mu^*_{\Omega,\lambda}$ and A_Ω are both bounded on $L^p(\mathbb{H})$ for $p \in [2, \infty)$.*

Proof. We first show the following fact: For any measurable function ϕ, we have

$$\int_{\mathbb{H}} \left(\mu^*_{\Omega,\lambda}(f)(x)\right)^2 \phi(x) dx \le C_{\lambda,\gamma} \int_{\mathbb{H}} \left(\mu_\Omega(f)(x)\right)^2 M\phi(x) dx, \tag{4.1}$$

where M denotes the Hardy-Littlewood maximal operator on $\mathbb{H}$.

In fact, without loss of generality, we may assume that $\phi \geq 0$, then

$$\begin{aligned}\int_{\mathbb{H}} \left(\mu^*_{\Omega,\lambda}(f)(x)\right)^2 \phi(x)dx &= \int_{\mathbb{H}} \iint_{\mathbb{H}\times\mathbb{R}_+} \left(\frac{t}{t+r(y^{-1}x)}\right)^{\gamma\lambda} |f*\Psi_t(y)|^2 \frac{dydt}{t^{\gamma+1}}\phi(x)dx \\ &\leq \int_{\mathbb{H}} \int_0^\infty |f*\Psi_t(y)|^2 \frac{dt}{t} \sup_{t>0} \left(\int_{\mathbb{H}} \left(\frac{t}{t+r(y^{-1}x)}\right)^{\gamma\lambda} \phi(x)\frac{dx}{t^\gamma}\right) dy \\ &\leq C_{\lambda,\gamma} \int_{\mathbb{H}} \left(\mu_\Omega(f)(y)\right)^2 M\phi(y)dy.\end{aligned}$$

Let $\phi \equiv 1$ in (4.1). Then by L^∞ boundedness of the Hardy-Littlewood maximal operator M we have

$$\int_{\mathbb{H}} \left(\mu^*_{\Omega,\lambda}(f)(x)\right)^2 dx \leq C_{\lambda,\gamma} \int_{\mathbb{H}} \left(\mu_\Omega(f)(x)\right)^2 dx.$$

Thus, the operator $\mu^*_{\Omega,\lambda}$ is bounded on $L^2(\mathbb{H})$. When $2 < p < \infty$, let $q = (p/2)'$. Then there is a non-negative function $\phi \in L^q(\mathbb{H})$ with $\|\phi\|_q \leq 1$ such that

$$\|\mu^*_{\Omega,\lambda}(f)\|_p^2 = \int_{\mathbb{H}} \left(\mu^*_{\Omega,\lambda}(f)(x)\right)^2 \phi(x)dx. \tag{4.2}$$

By (4.1), (4.2), Hölder's inequality and the $L^q\,(1 < q < \infty)$ boundedness of the maximal operator M we get

$$\|\mu^*_{\Omega,\lambda}(f)\|_p^2 \leq C_{\lambda,\gamma} \int_{\mathbb{H}} \left(\mu_\Omega(f)(x)\right)^2 M\phi(x)dx \leq C_{\lambda,\gamma}\|\mu_\Omega(f)\|_p^2 \|M\phi\|_q \leq C_{\lambda,p}\|\mu_\Omega(f)\|_p^2,$$

which gives the $L^p\,(2 \leq p < \infty)$ boundedness of $\mu^*_{\Omega,\lambda}$.

Since $A_\Omega(f)(x) \leq C_{\lambda,\gamma}\mu^*_{\Omega,\lambda}(f)(x)$ for any $x \in \mathbb{H}$, A_Ω is also bounded on $L^p(\mathbb{H})$ for $2 \leq p < \infty$. Hence we complete the proof of Theorem 4. □

Secondly, if Ψ is as in (1.6) and $1 < q < \infty$, we define the Littlewood-Paley operators $\mu_{\Omega,q}$, $A_{\Omega,q}$ and $\mu^*_{\Omega,\lambda,q}$ by

$$\mu_{\Omega,q}(f)(x) = \left(\int_0^\infty |f*\Psi_t(x)|^q \frac{dt}{t}\right)^{1/q},$$

$$A_{\Omega,q}(f)(x) = \left(\iint_{\Gamma(x)} |f*\Psi_t(y)|^q \frac{dydt}{t^{\gamma+1}}\right)^{1/q}$$

and

$$\mu^*_{\Omega,\lambda,q}(f)(x) = \left(\iint_{\mathbb{H}\times\mathbb{R}_+} \left(\frac{t}{t+r(y^{-1}x)}\right)^{\gamma\lambda} |f*\Psi_t(y)|^q \frac{dydt}{t^{\gamma+1}}\right)^{1/q}, \quad \lambda > 1,$$

respectively. Obviously, $\mu_{\Omega,q}$, $A_{\Omega,q}$ and $\mu^*_{\Omega,\lambda,q}$ are just the Littlewood-Paley operators discussed previously when $q = 2$.

Theorem 5. *Suppose that $2 \leq q < \infty$ and $\Omega \in L(\log L)^{1/2}(\Sigma)$ with the condition (1.5). Then we have the following conclusions:*

(1) *$\mu_{\Omega,q}$ is bounded on $L^p(\mathbb{H})$ for $1 < p < \infty$.*
(2) *$\mu^*_{\Omega,\lambda,q}$ and $A_{\Omega,q}$ are both bounded on $L^p(\mathbb{H})$ for $p \in [q, \infty)$.*

In fact, note that

$$\mu_{\Omega,q}(f)(x) \le (M_\Psi(|f|)(x))^{(q-2)/q}(S_\Psi(f)(x))^{2/q}$$

for any $x \in \mathbb{H}$. By a proof similar to that of Lemma 4 we can easily see L^p boundedness of M_Ψ. Thus, Theorem 1 and Hölder's inequality imply the conclusion (1) of Theorem 5.

The conclusion (2) of Theorem 5 is a direct consequence of the conclusion (1), as can be seen from the proof of Theorem 4.

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