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MAXIMAL SINGULAR INTEGRALS ON PRODUCT HOMOGENEOUS GROUPS

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Abstract. We prove L^p boundedness for $p \in (1, \infty)$ of maximal singular integral operators with rough kernels on product homogeneous groups under a sharp integrability condition of the kernels.

1. Introduction

Let $\mathbb{R}^d$, $d \geq 2$, be the d-dimensional Euclidean space. We assume that $\mathbb{R}^d$ is also equipped with a homogeneous group structure, where multiplication is given by a polynomial mapping; the underlying manifold is $\mathbb{R}^d$ itself. We also write $\mathbb{R}^d = \mathbb{H}$. Thus, $\mathbb{H}$ is associated with a dilation group $\{A_t\}_{t>0}$ of automorphisms of the group structure such that

$$A_t x = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_d} x_d), \quad x = (x_1, \dots, x_d) \in \mathbb{H},$$

where real numbers $a_1, \dots, a_d$ satisfy $0 < a_1 \leq a_2 \leq \dots \leq a_d$ (see [10], [21], [18], [6] and [11, Section 2 of Chapter 1]). So, we have, for each $t > 0$,

$$A_t(xy) = (A_t x)(A_t y), \quad x, y \in \mathbb{H}.$$

Consequently, $\mathbb{H}$ is endowed with both the Euclidean structure and a homogeneous nilpotent Lie group structure. The group law of $\mathbb{H}$ is given by a polynomial mapping which conforms to the Campbell-Hausdorff formula in a corresponding Lie algebra via an exponential map and the action of the automorphism family $\{A_t\}$. We note that the identity is the origin 0 and $x^{-1} = -x$; furthermore, Lebesgue measure is bi-invariant Haar measure.

Let us recall a norm function $r(x)$ associated with $\{A_t\}$. The function $r(x)$, which is non-negative and vanishes only at the origin, satisfies that $r(A_t x) = t r(x)$ for $t > 0$ and $x \in \mathbb{R}^d$. We assume that $r(x)$ is even, continuous on $\mathbb{R}^d$ and smooth in $\mathbb{R}^d \setminus \{0\}$, and also that the unit sphere $\Sigma_d = \{x \in \mathbb{R}^d : r(x) = 1\}$ defined by the norm function coincides with the unit sphere $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, where $|x|$ denotes the Euclidean norm. Let $\gamma = a_1 + \dots + a_d$ (the homogeneous dimension of $\mathbb{H}$). We shall

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use the formula:

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_0^\infty \int_{\Sigma_d} f(A_t\theta)t^{\gamma-1}\,dS_d(\theta)\,dt, \quad dS_d = \omega\,d\sigma_d,$$

where ω is a strictly positive C^∞ function on Σ_d and $d\sigma_d$ is the Lebesgue surface measure on Σ_d. See [18, 6] and also [3, 10, 11, 14, 19, 20, 21] for more details and related results.

We consider a function Ω which is locally integrable in $\mathbb{R}^d \setminus \{0\}$ and homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, that is, $\Omega(A_t x) = \Omega(x)$ for $x \neq 0$, $t > 0$. We assume the cancellation property:

$$\int_{\Sigma_d} \Omega(\theta)\,dS_d(\theta) = 0. \tag{1.1}$$

Convolution on $\mathbb{H}$ is defined by

$$f * g(x) = \int_{\mathbb{H}} f(y)g(y^{-1}x)\,dy.$$

Let

$$Tf(x) = \text{p.v.} f * K(x) = \text{p.v.} \int_{\mathbb{H}} f(y)K(y^{-1}x)\,dy \tag{1.2}$$

for appropriate functions f, where $K(x) = \Omega(x')r(x)^{-\gamma}$, $x' = A_{r(x)^{-1}}x$ for $x \neq 0$. We also define the maximal singular integral operator

$$T_* f(x) = \sup_{\epsilon>0} \left| \int_{r(y)>\epsilon} f(xy^{-1})K(y)\,dy \right|. \tag{1.3}$$

Then the following results are known.

Theorem A ([21]). *If $\Omega \in L\log L(\Sigma_d)$ with (1.1) and Tf is as in (1.2), then T is bounded on $L^p(\mathbb{H})$ for all $p \in (1,\infty)$.*

Theorem B ([18]). *Let $T_* f$ be defined as in (1.3) with $\Omega \in L\log L(\Sigma_d)$ satisfying (1.1). Let $p \in (1,\infty)$. Then the operator T_* is bounded on $L^p(\mathbb{H})$.*

We refer to [4, 12, 13, 14, 15, 16] for relevant results.

Part of a theory of Duoandikoetxea and Rubio de Francia [8] for singular integrals on the Euclidean spaces has been generalized to the case of homogeneous groups by [18]. The arguments of [18] replace Fourier transform estimates by a variant of Tao's L^2 estimates via convolution (see [21]). As a result, [18] proved Theorem B and some weighted estimates, and also gave another proof of Theorem A.

Also, it has been shown in [6] that the theory of [18] extends to the case of product homogeneous groups to treat multiple singular integrals. Let $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a product homogeneous group with $\mathbb{R}^{n_1} = \mathbb{H}_1$, $\mathbb{R}^{n_2} = \mathbb{H}_2$, where $n = n_1 + n_2$ and $\mathbb{H}_1$, $\mathbb{H}_2$ are homogeneous groups with dilations $A_t^{(1)}$, $A_t^{(2)}$ and norm functions r_1, r_2, respectively. We consider a function Ω in $L^1(\Sigma_{n_1} \times \Sigma_{n_2})$ which satisfies

$$\int_{\Sigma_{n_1}} \Omega(u,v)\, dS_{n_1}(u) = 0, \quad \text{for all } v \in \Sigma_{n_2}, \tag{1.4}$$

$$\int_{\Sigma_{n_2}} \Omega(u,v)\, dS_{n_2}(v) = 0, \quad \text{for all } u \in \Sigma_{n_1}. \tag{1.5}$$

Define

$$K(u,v) = r_1(u)^{-\gamma_1} r_2(v)^{-\gamma_2} \Omega(u', v'), \quad u' = A^{(1)}_{r_1(u)^{-1}} u, v' = A^{(2)}_{r_2(v)^{-1}} v,$$

where γ_1 and γ_2 are the homogeneous dimensions of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. We consider the multiple singular integral

$$Tf(x,y) = \text{p.v. } f * K(x,y) = \text{p.v.} \int_{\mathbb{H}_1 \times \mathbb{H}_2} f(xu^{-1}, yv^{-1}) K(u,v)\, du\, dv. \tag{1.6}$$

Then the following result is proved in [6].

Theorem C. *Let T be defined as in (1.6) with Ω in $L(\log L)^2(\Sigma_{n_1} \times \Sigma_{n_2})$ satisfying (1.4) and (1.5). Let $1 < p < \infty$. The operator T is then bounded on $L^p(\mathbb{H}_1 \times \mathbb{H}_2)$.*

We can find in [2] the optimality of the $L(\log L)^2$ integrability condition for multiple singular integrals with Euclidean convolution.

Let us recall that the maximal singular integral is defined by

$$T_* f(x,y) = \sup_{\substack{\epsilon_1 > 0,\\ \epsilon_2 > 0}} \left| \int_{\substack{r_1(u) > \epsilon_1,\\ r_2(v) > \epsilon_2}} f(xu^{-1}, yv^{-1}) K(u,v)\, du\, dv \right|. \tag{1.7}$$

In this note we shall prove the following.

Theorem 1. *Let T_* be defined as in (1.7). Suppose that Ω is in $L(\log L)^2(\Sigma_{n_1} \times \Sigma_{n_2})$ and satisfies (1.4), (1.5). Then T_* is bounded on $L^p(\mathbb{H}_1 \times \mathbb{H}_2)$ for all $p \in (1, \infty)$.*

Previous works concerning singular integrals on product of Euclidean spaces can be found in [1, 2, 7, 9]. Theorem 1 is an analogue of a result of [2] for multiple singular integrals on product homogeneous groups.

Similarly to the proof of Theorem C in [6], we use extrapolation arguments in proving Theorem 1 by applying the following result.

Theorem 2. *Let $1 < s \le 2$. Suppose that Ω is in $L^s(\Sigma_{n_1} \times \Sigma_{n_2})$ and satisfies (1.4), (1.5). Then, for $1 < p < \infty$ we have*

$$\|T_* f\|_p \le C_p (s-1)^{-2} \|\Omega\|_s \|f\|_p$$

with a constant C_p independent of s and Ω.

Let Ω be as in Theorem 1. Then there exist a sequence $\{\Omega_k\}$ of functions in $L^1(\Sigma_{n_1}\times\Sigma_{n_2})$ and a sequence $\{c_k\}$ of non-negative real numbers such that each Ω_k satisfies (1.4) and (1.5), $\sup_{k\geq 1}\|\Omega_k\|_{1+1/k}\leq 1$, $\sum_{k=1}^\infty k^2 c_k<\infty$, and

$$\Omega=\sum_{k=1}^\infty c_k\Omega_k.$$

Theorem 1 easily follows from this decomposition of Ω and Theorem 2 (see [17, 15, 16]).

In Section 2 we recall some preliminary results from [6]. We shall prove Theorem 2 in Section 3 by using results of [6, 5, 18]; similar arguments, via Fourier transform estimates, for singular integrals with Euclidean convolution can be found in [1, 2].

2. Preliminaries

Let $\rho\geq 2$ and let $\psi_j\in C_0^\infty(\mathbb{R})$, $j\in\mathbb{Z}$ (the set of integers), be such that

$$\operatorname{supp}(\psi_j)\subset\{t\in\mathbb{R}:\rho^j\leq t\leq\rho^{j+2}\},\quad \psi_j\geq 0,$$
$$(\log 2)\sum_{j\in\mathbb{Z}}\psi_j(t)=1\quad\text{for } t>0;$$

furthermore,

$$|(d/dt)^m\psi_j(t)|\leq c_m|t|^{-m}\quad\text{for } m=0,1,2,\dots,$$

where c_m is a constant independent of ρ; we note that this is possible since $\rho\geq 2$.

Suppose that F belongs to $L^1(\mathbb{H}_1\times\mathbb{H}_2)$ with support in D_0, where $D_0=D_0^{(1)}\times D_0^{(2)}$,

$$D_0^{(1)}=\{x\in\mathbb{H}_1:1\leq r_1(x)\leq 2\},\quad D_0^{(2)}=\{y\in\mathbb{H}_2:1\leq r_2(y)\leq 2\}.$$

Let $\delta_s^{(1)}f(x)=s^{-\gamma_1}f((A_s^{(1)})^{-1}x)$, $\delta_t^{(2)}g(y)=t^{-\gamma_2}g((A_t^{(2)})^{-1}y)$. Define $\delta_{s,t}=\delta_s^{(1)}\otimes\delta_t^{(2)}$ and let

$$S_{j,k}F(x,y)=\int_0^\infty\int_0^\infty\psi_j(s)\psi_k(t)\delta_{s,t}F(x,y)\,\frac{ds}{s}\,\frac{dt}{t}. \tag{2.1}$$

Then $\sum_{j,k\in\mathbb{Z}}S_{j,k}K_0=K$, where

$$K_0(x,y)=\begin{cases}K(x,y), & (x,y)\in D_0,\\ 0, & \text{otherwise.}\end{cases} \tag{2.2}$$

For $s\geq 1$, let $L^s(D_0)$ denote the subspace of $L^s(\mathbb{H}_1\times\mathbb{H}_2)$ consisting of functions F with support in D_0. Define

$$M_Ff(x,y)=\sup_{j,k\in\mathbb{Z}}|f*S_{j,k}(|F|)(x,y)|$$

for $F \in L^s(D_0)$. The following result is Lemma 8 of [6].

Lemma 1. *Let $p > 1$. Suppose that $s \in (1, 2], \rho = 2^{s'}$, $s' = s/(s-1)$, and $F \in L^s(D_0)$. Then,*

$$\|M_F f\|_p \le C_p (s-1)^{-2} \|F\|_s \|f\|_p$$

with a positive constant C_p independent of s and F.

Also, we need another result of [6].

Lemma 2. *Let $1 < s \le 2$. Suppose that Ω belongs to $L^s(\Sigma_{n_1} \times \Sigma_{n_2})$ and satisfies (1.4) and (1.5). Let*

$$Rf(x, y) = \sup_{\ell, m \in \mathbb{Z}} \left| \sum_{j=\ell}^{\infty} \sum_{k=m}^{\infty} f * S_{j,k} K_0(x, y) \right|,$$

where K_0 is defined by (2.2) and $S_{j,k}K_0$ is as in (2.1) with K_0 in place of F. Let $\rho = 2^{s'}$. Then, for $p \in (1, \infty)$ there exists a positive constant C_p independent of $s \in (1, 2]$ and $\Omega \in L^s$ such that

$$\|Rf\|_p \le C_p A(s, \Omega) \|f\|_p,$$

where $A(s, \Omega) = (s-1)^{-2} \|\Omega\|_s$.

This is Proposition 2 of [6].

3. Proof of Theorem 2.

We first note that

$$S_{j,k} K_0(x, y) = r_1(x)^{-\gamma_1} r_2(y)^{-\gamma_2} \Omega(x', y') \int_{1/2}^{1} \psi_j(s r_1(x)) \, \frac{ds}{s} \int_{1/2}^{1} \psi_k(t r_2(y)) \, \frac{dt}{t},$$

where $x' = A^{(1)}_{r_1(x)^{-1}} x$, $y' = A^{(2)}_{r_2(y)^{-1}} y$. From this we easily see that

$$\operatorname{supp}(S_{j,k} K_0) \subset \left\{ \rho^j \le r_1(x) \le 2\rho^{j+2} \right\} \times \left\{ \rho^k \le r_2(y) \le 2\rho^{k+2} \right\}.$$

Thus, if $\ell, m \in \mathbb{Z}$ are determined by the conditions $\rho^{\ell+2} \le \epsilon < \rho^{\ell+3}$, $\rho^{m+2} \le \delta < \rho^{m+3}$ and if f is a compactly supported smooth function, we have

$$\begin{aligned} (3.1) \qquad & \int_{\substack{r_1(u) > \epsilon, \\ r_2(v) > \delta}} f(xu^{-1}, yv^{-1}) K(u, v) \, du \, dv \\ &= \sum_{\substack{j \ge \ell, \\ k \ge m}} \int_{\substack{r_1(u) > \epsilon, \\ r_2(v) > \delta}} f(xu^{-1}, yv^{-1}) S_{j,k} K_0(u, v) \, du \, dv \\ &= A_{\epsilon,\delta} f(x, y) + B_{\epsilon,\delta} f(x, y) + C_{\epsilon,\delta} f(x, y) + D_{\epsilon,\delta} f(x, y), \end{aligned}$$

where

$$A_{\epsilon,\delta}f(x,y) = \sum_{\substack{j>\ell+3,\\ k>m+3}} \int_{\mathbb{H}_1\times\mathbb{H}_2} f(xu^{-1}, yv^{-1}) S_{j,k}K_0(u,v)\,du\,dv$$
$$= \sum_{\substack{j>\ell+3,\\ k>m+3}} f * S_{j,k}K_0(x,y),$$
$$B_{\epsilon,\delta}f(x,y) = \sum_{\substack{\ell+3\ge j\ge \ell,\\ k>m+3}} \int_{\{r_1(u)>\epsilon\}\times\mathbb{H}_2} f(xu^{-1}, yv^{-1}) S_{j,k}K_0(u,v)\,du\,dv,$$
$$C_{\epsilon,\delta}f(x,y) = \sum_{\substack{j>\ell+3,\\ m+3\ge k\ge m}} \int_{\mathbb{H}_1\times\{r_2(v)>\delta\}} f(xu^{-1}, yv^{-1}) S_{j,k}K_0(u,v)\,du\,dv,$$
$$D_{\epsilon,\delta}f(x,y) = \sum_{\substack{\ell+3\ge j\ge \ell,\\ m+3\ge k\ge m}} \int_{\substack{r_1(u)>\epsilon,\\ r_2(v)>\delta}} f(xu^{-1}, yv^{-1}) S_{j,k}K_0(u,v)\,du\,dv.$$

Let

$$A_*f(x,y) = \sup_{\epsilon,\delta>0} |A_{\epsilon,\delta}f(x,y)|, \qquad B_*f(x,y) = \sup_{\epsilon,\delta>0} |B_{\epsilon,\delta}f(x,y)|,$$
$$C_*f(x,y) = \sup_{\epsilon,\delta>0} |C_{\epsilon,\delta}f(x,y)|, \qquad D_*f(x,y) = \sup_{\epsilon,\delta>0} |D_{\epsilon,\delta}f(x,y)|.$$

Then, (3.1) implies

$$T_*f(x,y) \le A_*f(x,y) + B_*f(x,y) + C_*f(x,y) + D_*f(x,y). \tag{3.2}$$

Let $\rho = 2^{s'}$. Since $A_*f \le Rf$, by Lemma 2 we have

$$\|A_*f\|_p \le C_p A(s,\Omega)\|f\|_p. \tag{3.3}$$

Also, since $D_*f \le CM_{K_0}(|f|)$, Lemma 1 implies

$$\|D_*f\|_p \le C\|M_{K_0}(|f|)\|_p \le C_p A(s,\Omega)\|f\|_p. \tag{3.4}$$

To estimate B_*f, we note that

$$|B_{\epsilon,\delta}f(x,y)|$$
$$\le \sum_{\ell+3\ge j\ge\ell} \int_{\rho^\ell\le r_1(u)\le 2\rho^{\ell+5}} \left| \sum_{k>m+3} \int_{\mathbb{H}_2} f(xu^{-1}, yv^{-1}) S_{j,k}K_0(u,v)\,dv \right| du.$$

By changing variables with respect to u, we see that the right hand side is equal to

$$\sum_{\ell+3\ge j\ge\ell} \int_{\rho^\ell}^{2\rho^{\ell+5}} \int_{\Sigma_{n_1}} \left| \sum_{k>m+3} F_k(x,y,s,\theta) \right| \Psi_j(s)\,\frac{ds}{s}\,dS_{n_1}(\theta),$$

where

$$F_k(x,y,s,\theta) = \int_{\mathbb{H}_2} f(x(A_s^{(1)}\theta)^{-1}, yv^{-1})\Omega(\theta,v')r_2(v)^{-\gamma_2}\Psi_k(r_2(v))\,dv,$$

$\Psi_k(t) = \int_{1/2}^1 \psi_k(rt)\, dr/r$. Thus, since $0 \le \Psi_j(s) \le 1$, we have

$$|B_{\epsilon,\delta} f(x,y)| \le C \int_{\rho^\ell}^{2\rho^{\ell+5}} \int_{\Sigma_{n_1}} \left| \sum_{k>m+3} F_k(x,y,s,\theta) \right| \frac{ds}{s}\, dS_{n_1}(\theta).$$

We write

$$K_\theta^{(2)}(v) = K_0(\theta, v), \quad S_k^{(2)} K_\theta^{(2)}(v) = \int_0^\infty \psi_k(t) \delta_t^{(2)} K_\theta^{(2)}(v)\, \frac{dt}{t}.$$

Then

$$F_k(x,y,s,\theta) = f(x(A_s^{(1)}\theta)^{-1}, \cdot) *_{(2)} S_k^{(2)} K_\theta^{(2)}(y),$$

where $*_{(2)}$ denotes the convolution on $\mathbb{H}_2$. Consequently,

$$(3.5) \quad |B_{\epsilon,\delta} f(x,y)| \le C \int_{\rho^\ell}^{2\rho^{\ell+5}} \int_{\Sigma_{n_1}} \left| \sum_{k>m+3} f(x(A_s^{(1)}\theta)^{-1}, \cdot) *_{(2)} S_k^{(2)} K_\theta^{(2)}(y) \right| \frac{ds}{s}\, dS_{n_1}(\theta).$$

Let

$$R_\theta^{(2)} g(y) = \sup_{m\in\mathbb{Z}} \left| \sum_{k>m} g *_{(2)} S_k^{(2)} K_\theta^{(2)}(y) \right|$$

for g on $\mathbb{H}_2$. We write $f_x(y) = f(x,y)$ when considering $f(x,y)$ as a function on $\mathbb{H}_2$ fixing x; similarly, we write $f_y(x) = f(x,y)$. Define

$$F^\theta(x,y) = R_\theta^{(2)} f_x(y).$$

Then, using (3.5), we have

$$(3.6) \qquad |B_* f(x,y)| \le C \sup_{\ell\in\mathbb{Z}} \int_{\rho^\ell}^{2\rho^{\ell+5}} \int_{\Sigma_{n_1}} F^\theta(x(A_s^{(1)}\theta)^{-1}, y)\, \frac{ds}{s}\, dS_{n_1}(\theta)$$
$$\le C \log\rho \int_{\Sigma_{n_1}} M_\theta^{(1)} F_y^\theta(x)\, dS_{n_1}(\theta),$$

where

$$M_\theta^{(1)} h(x) = \sup_{t>0} \frac{1}{t} \int_0^t |h(x(A_s^{(1)}\theta)^{-1})|\, ds$$

for h on $\mathbb{H}_1$. The last inequality of (3.6) can be seen as follows. Take a positive integer d such that $2^d \le 2\rho^5 < 2^{d+1}$. Then

$$\begin{aligned} \int_{\rho^\ell}^{2\rho^{\ell+5}} |h(x(A_s^{(1)}\theta)^{-1})|\, \frac{ds}{s} &\le \sum_{i=0}^d \int_{2^i\rho^\ell}^{2^{i+1}\rho^\ell} |h(x(A_s^{(1)}\theta)^{-1})|\, \frac{ds}{s} \\ &\le \sum_{i=0}^d 2M_\theta^{(1)} h(x) = 2(d+1) M_\theta^{(1)} h(x) \\ &\le C(\log\rho) M_\theta^{(1)} h(x), \end{aligned}$$

since $d \sim \log\rho$. This implies what we need.

By M. Christ [5] $M_\theta^{(1)}$ is bounded on L^p, $p > 1$, with a bound independent of θ. Thus, using (3.6) and the Minkowski inequality we have

$$\|B_* f\|_p \le C(\log \rho) \int_{\Sigma_{n_1}} \|F^\theta\|_p \, dS_{n_1}(\theta). \tag{3.7}$$

By Lemma 9 of [18] with $\rho = 2^{s'}$, we have

$$\|R_\theta^{(2)} g\|_p \le C_p(\log \rho) \left(\int_{\Sigma_{n_2}} |\Omega(\theta, \omega)|^s \, dS_{n_2}(\omega) \right)^{1/s} \|g\|_p.$$

Thus

$$\|F_x^\theta\|_p \le C_p(\log \rho) \left(\int_{\Sigma_{n_2}} |\Omega(\theta, \omega)|^s \, dS_{n_2}(\omega) \right)^{1/s} \|f_x\|_p.$$

Using this in (3.7) and noting $\|F^\theta\|_p = (\int \|F_x^\theta\|_p^p \, dx)^{1/p}$, we have

$$\begin{aligned} \|B_* f\|_p &\le C_p(\log \rho)^2 \int_{\Sigma_{n_1}} \left(\int_{\Sigma_{n_2}} |\Omega(\theta, \omega)|^s \, dS_{n_2}(\omega) \right)^{1/s} dS_{n_1}(\theta) \|f\|_p \\ &\le C_p(\log \rho)^2 \|\Omega\|_s \|f\|_p, \end{aligned} \tag{3.8}$$

where the last inequality follows from Hölder's inequality.

Similarly, we have

$$\|C_* f\|_p \le C_p(\log \rho)^2 \|\Omega\|_s \|f\|_p. \tag{3.9}$$

Combining (3.2), (3.3), (3.4), (3.8) and (3.9), we get the conclusion of Theorem 2.

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