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Analysis 31, 273–291 (2011) / **DOI** 10.1524/anly.2011.1124

Singular integrals associated with functions of finite type and extrapolation

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Received: December 1, 2010

Summary: We consider a singular integral along a submanifold of finite type. We prove a certain L^p estimate for the singular integral, which is useful in applying an extrapolation method that shows L^p boundedness of the singular integral under a sharp condition of the kernel.

1 Introduction

Let $B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$ and let $\Phi : B(0,1) \to \mathbb{R}^d$ be a smooth function. We assume that Φ is of finite type at the origin, that is, for any $\xi \in S^{d-1}$ (the unit sphere in $\mathbb{R}^d$) there exists a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $|\alpha| \geq 1$ and $\partial_x^\alpha \langle \Phi(x), \xi \rangle|_{x=0} \neq 0$, where $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$.

Let a function Ω in $L^1(S^{n-1})$ satisfy

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0, \tag{1.1}$$

where $d\sigma$ denotes the Lebesgue surface measure on the unit sphere S^{n-1} in $\mathbb{R}^n$. Throughout this note we assume $n \geq 2$. Let Δ_s, $s \geq 1$, denote the collection of functions h on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ satisfying

$$\|h\|_{\Delta_s} = \sup_{j \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s \, dt/t \right)^{1/s} < \infty,$$

where $\mathbb{Z}$ denotes the set of integers. We define

$$\omega(h,t) = \sup_{|s| < tR/2} \int_R^{2R} |h(r-s) - h(r)|\, dr/r, \qquad t \in (0,1],$$

where the supremum is taken over all s and R such that $|s| < tR/2$ (see [6, 12]). For $\eta > 0$, let Λ^η denote the family of functions h satisfying

$$\|h\|_{\Lambda^\eta} = \sup_{t \in (0,1]} t^{-\eta} \omega(h,t) < \infty.$$

AMS 2010 subject classification: Primary: 42B20, 42B25
Key words and phrases: Singular integral, singular Radon transform, submanifold of finite type, extrapolation

Define a space $\Lambda_s^\eta = \Delta_s \cap \Lambda^\eta$ and set $\|h\|_{\Lambda_s^\eta} = \|h\|_{\Delta_s} + \|h\|_{\Lambda^\eta}$ for $h \in \Lambda_s^\eta$.

We consider a singular Radon transform of the form:

$$T(f)(x) = \text{p.v.} \int_{B(0,1)} f(x - \Phi(y))K(y)\, dy \tag{1.2}$$
$$= \lim_{\epsilon \to 0} \int_{1>|y|>\epsilon} f(x - \Phi(y))K(y)\, dy$$

for an appropriate function f on $\mathbb{R}^d$, where $K(y) = h(|y|)\Omega(y')|y|^{-n}$, $y' = |y|^{-1}y$, $h \in \Delta_1$. See Stein [13], Fan, Guo, and Pan [4], Al-Salman and Pan [1] and also [2, 5, 14] for this singular integral and related topics.

In the previous works, the operator T was studied under the condition that h is a constant function. In this note, we consider the operator T under a more general condition on h. We shall prove the following:

Theorem 1.1 *Let $q \in (1, 2]$, $\Omega \in L^q(S^{n-1})$ and $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that Ω satisfies the condition* (1.1)*. Let T be defined as in* (1.2)*. Then we have*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \le C_p(q-1)^{-1}\|h\|_{\Lambda_1^\eta}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q, h and Ω.

Let $L\log L(S^{n-1})$ denote the Zygmund class of the functions F on S^{n-1} satisfying

$$\int_{S^{n-1}} |F(\theta)| \log(2 + |F(\theta)|)\, d\sigma(\theta) < \infty.$$

Then, as an application of Theorem 1.1 and extrapolation, we have the following theorem.

Theorem 1.2 *Let $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that Ω is in $L\log L(S^{n-1})$ and satisfies the condition* (1.1)*. Let T be as in* (1.2)*. Then we have*

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \le C_p\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$.

The extrapolation argument that proves Theorem 1.2 from Theorem 1.1 can be found in [8, 9, 10, 11] (see also [15, Chap. XII, pp. 119–120]). If the function h is assumed to be a constant function in Theorem 1.2, we have a result of Al-Salman and Pan shown in [1] (see [1, Theorem 1.1]); so we can give a different proof of the result by applying Theorem 1.1 and extrapolation. Relevant results can be found in [8, 9, 10, 11].

In Section 2, we shall prove Theorem 1.1. Consider a singular integral of the form

$$S(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y))h(|y|)\Omega(y')|y|^{-n}\, dy,$$

where $P(y)$ is a polynomial mapping from $\mathbb{R}^n$ to $\mathbb{R}^d$ satisfying $P(-y) = -P(y)$ ($P \neq 0$), $h \in \Delta_s$ for $s \in (1, 2]$ and Ω is a function in $L^q(S^{n-1})$, $q \in (1, 2]$, satisfying (1.1). Then, it has been proved that

$$\|S(f)\|_{L^p(\mathbb{R}^d)} \le C_p(q-1)^{-1}(s-1)^{-1}\|\Omega\|_{L^q(S^{n-1})}\|h\|_{\Delta_s}\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q, s, Ω, h and the polynomial components of P if they are of fixed degree (see [8, Theorem 1]). Outline of our proof of Theorem 1.1 is similar to that of the proof for [8, Theorem 1]. We apply methods of [4] to obtain some basic estimates. We need to assume that $h \in \Lambda_1^\eta$ for some $\eta > 0$ to prove certain Fourier transform estimates. As in [8] (see also [9, 10]), a key idea of the proof of Theorem 1.1 is to apply a Littlewood–Paley decomposition adapted to an appropriate lacunary sequence depending on q for which $\Omega \in L^q(S^{n-1})$.

In Section 3, we shall give analogs of Theorems 1.1 and 1.2 for a maximal singular integral operator related to T. In what follows we also write $\|f\|_{L^p(\mathbb{R}^d)} = \|f\|_p$ and $\|\Omega\|_{L^q(S^{n-1})} = \|\Omega\|_q$. Throughout this note, the letter C will be used to denote non-negative constants which may be different in different occurrences.

2 Proof of Theorem 1.1

Let M be a positive integer. We write $\Phi(y) = (\Phi_1(y), \dots, \Phi_d(y))$. Let $P_j(y)$ be the Taylor polynomial of $\Phi_j(y)$ at the origin defined by

$$P_j(y) = \sum_{|\alpha| \le M-1} \frac{1}{\alpha!} (\partial_y^\alpha \Phi_j)(0) y^\alpha,$$

where $\alpha! = \alpha_1! \dots \alpha_n!$ and $y^\alpha = y_1^{\alpha_1} \dots y_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $y = (y_1, \dots, y_n)$. We write $P(y) = (P_1(y), P_2(y), \dots, P_d(y))$ and

$$P(y) = \sum_{j=1}^{\ell} Q_j(y), \qquad Q_j(y) = \sum_{|\gamma| = N(j)} a_\gamma y^\gamma \quad (a_\gamma \in \mathbb{R}^d),$$

where $0 = N(1) < N(2) < \dots < N(\ell)$, $Q_j \neq 0$ for $j \ge 2$. Let $\beta_m = \rho^{N(m)}$ and $\alpha_m = \tau(q-1)/(qN(m))$ for $2 \le m \le \ell$, where $\tau = 4^{-1}\min(1, \eta)$, $\rho \ge 2$. Also, let $\beta_{\ell+1} = \rho^M$ and $\alpha_{\ell+1} = \epsilon_0(q-1)/q$ for some $\epsilon_0 \in (0, 1/4)$. The positive integer M and the positive number ϵ_0 will be specified in the following (see Lemma 2.4 below).

Let T be as in Theorem 1.1. Let $E_k = \{x \in \mathbb{R}^n : \rho^k \le |x| < \rho^{k+1}\}$, $k \in \mathbb{Z}$, $\rho \ge 2$. Then $T(f)(x) = \sum_{k=-\infty}^{-1} \sigma_k * f(x)$, where $\{\sigma_k\}_{k=-\infty}^{-1}$ is a sequence of Borel measures on $\mathbb{R}^d$ such that

$$\sigma_k * f(x) = \int_{E_k} f(x - \Phi(y)) K(y)\, dy. \tag{2.1}$$

Put $P^{(m)}(y) = \sum_{j=1}^m Q_j(y)$ for $m = 1, 2, \dots, \ell$ and $P^{(\ell+1)}(y) = \Phi(y)$. Consider a sequence $\mu^{(m)} = \{\mu_k^{(m)}\}_{k=-\infty}^{-1}$ of positive measures on $\mathbb{R}^d$ such that

$$\mu_k^{(m)} * f(x) = \int_{E_k} f\left(x - P^{(m)}(y)\right) |K(y)|\, dy$$

for $m = 1, 2, \dots, \ell+1$. Note that $\mu_k^{(1)} = (\int_{E_k} |K(y)|\, dy)\delta_{P(0)}$, where δ_a is Dirac's delta function on $\mathbb{R}^d$ concentrated at a. Let $\sigma^{(m)} = \{\sigma_k^{(m)}\}_{k=-\infty}^{-1}$ be a sequence of Borel measures on $\mathbb{R}^d$ such that

$$\sigma_k^{(m)} * f(x) = \int_{E_k} f\left(x - P^{(m)}(y)\right) K(y)\, dy,$$

for $m = 1, 2, \ldots, \ell + 1$. We note that $\sigma_k^{(1)} = 0$ by (1.1) and

$$(\sigma_k^{(m)} * f)\hat{\ }(\xi) = \hat{f}(\xi) \int_{E_k} e^{-2\pi i \langle P^{(m)}(y), \xi\rangle} K(y)\, dy,$$

where $\hat{f}$ denotes the Fourier transform of f. A similar formula holds for $\mu_k^{(m)}$.

Let $\{\gamma(j,k)\}_{k=1}^{r_j}$ be an enumeration of $\{\gamma\}_{|\gamma|=N(j)}$ for $1 \le j \le \ell$. Define a linear mapping L_j from $\mathbb{R}^d$ to $\mathbb{R}^{r_j}$ by

$$L_j(\xi) = (\langle a_{\gamma(j,1)}, \xi\rangle, \langle a_{\gamma(j,2)}, \xi\rangle, \ldots, \langle a_{\gamma(j,r_j)}, \xi\rangle),$$

for $1 \le j \le \ell$. Let $L_{\ell+1}$ be the identity mapping on $\mathbb{R}^d$. Let $s_j = \operatorname{rank} L_j$. For $j \ge 2$, there exist non-singular linear transformations $R_j : \mathbb{R}^d \to \mathbb{R}^d$ and $H_j : \mathbb{R}^{s_j} \to \mathbb{R}^{s_j}$ such that

$$|H_j \pi_{s_j}^d R_j(\xi)| \le |L_j(\xi)| \le C|H_j \pi_{s_j}^d R_j(\xi)|,$$

where $\pi_{s_j}^d(\xi) = (\xi_1, \ldots, \xi_{s_j})$ is the projection and C is a constant depending only on r_j (see [5]).

Let φ be a function in $C^\infty(\mathbb{R})$ satisfying $\varphi(r) = 1$ for $|r| < 1/2$ with support in $\{|r| \le 1\}$. Define a sequence $\tau^{(m)} = \{\tau_k^{(m)}\}_{k=-\infty}^{-1}$ of Borel measures by

$$\hat{\tau}_k^{(m)}(\xi) = \hat{\sigma}_k^{(m+1)}(\xi)\Phi_{k,m+1}(\xi) - \hat{\sigma}_k^{(m)}(\xi)\Phi_{k,m}(\xi) \tag{2.2}$$

for $m = 1, 2, \ldots, \ell$, where

$$\Phi_{k,m}(\xi) = \prod_{j=m+1}^{\ell+1} \varphi\left(\beta_j^k |H_j \pi_{s_j}^d R_j(\xi)|\right)$$

if $1 \le m \le \ell$ and $\Phi_{k,\ell+1} = 1$. Then $\sigma_k = \sigma_k^{(\ell+1)} = \sum_{m=1}^{\ell} \tau_k^{(m)}$. We note that

$$\Phi_{k,m+1}(\xi)\varphi\left(\beta_{m+1}^k |H_{m+1}\pi_{s_{m+1}}^d R_{m+1}(\xi)|\right) = \Phi_{k,m}(\xi) \quad (1 \le m \le \ell). \tag{2.3}$$

For $1 \le m \le \ell$, let $T_\rho^{(m)}(f) = \sum_{k=-\infty}^{-1} \tau_k^{(m)} * f$. Then $T = \sum_{m=1}^{\ell} T_\rho^{(m)}$.

For a sequence $\nu = \{\nu_k\}_{k=-\infty}^{-1}$ of finite Borel measures on $\mathbb{R}^d$, let $\nu^*(f)(x) = \sup_k ||\nu_k| * f(x)|$, where $|\nu_k|$ denotes the total variation. We consider the maximal operators $\left(\mu^{(m)}\right)^*$ $(1 \le m \le \ell + 1)$. We also write $\left(\mu^{(\ell+1)}\right)^* = \mu_\rho^*$.

Let $\theta \in (0, 1)$. For $p \in (1, \infty)$ let $p' = p/(p-1)$ and $\delta(p) = |1/p - 1/p'|$. Then we prove the following two propositions.

Proposition 2.1 *Let $p > 1 + \theta$ and $1 \le j \le \ell + 1$. Then we have*

$$\left\| (\mu^{(j)})^*(f) \right\|_{L^p(\mathbb{R}^d)} \le C(\log \rho) \|h\|_{\Lambda_1^\eta} \|\Omega\|_{L^q(S^{n-1})} B^{2/p} \|f\|_{L^p(\mathbb{R}^d)}, \tag{2.4}$$

where $B = \left(1 - \rho^{-\theta\kappa/q'}\right)^{-1}$ for some positive constant κ such that

$$\left(1 - \beta_m^{-\theta\alpha_m}\right)^{-1} \le B$$

for all m with $2 \le m \le \ell + 1$. The constant C is independent of $q \in (1, 2]$, $h \in \Lambda_1^\eta$, $\Omega \in L^q(S^{n-1})$ and ρ.

Proposition 2.2 *Let $p \in (1 + \theta, (1 + \theta)/\theta)$ and $1 \le m \le \ell$. Then*

$$\|T_\rho^{(m)}(f)\|_{L^p(\mathbb{R}^d)} \le C(\log \rho)\|h\|_{\Lambda_1^\eta} \|\Omega\|_{L^q(S^{n-1})} B^{1+\delta(p)} \|f\|_{L^p(\mathbb{R}^d)},$$

where B is as in Proposition 2.1 and the constant C is independent of $q \in (1, 2]$, $h \in \Lambda_1^\eta$, $\Omega \in L^q(S^{n-1})$ and ρ.

We can easily derive Theorem 1.1 from Proposition 2.2. Proposition 2.1 is used to prove Proposition 2.2. To prove Proposition 2.2 we also need the following.

Lemma 2.3 *Let $q \in (1, 2]$, $\Omega \in L^q(S^{n-1})$, $h \in \Lambda_1^\eta$ and $A = (\log \rho)\|h\|_{\Lambda_1^\eta} \|\Omega\|_q$. Let $\tau_k^{(m)}$ be as in (2.2). Then, for $1 \le m \le \ell$ we have*

$$\|\tau_k^{(m)}\| = |\tau_k^{(m)}|(\mathbb{R}^d) \le c_1 A, \tag{2.5}$$

$$|\hat{\tau}_k^{(m)}(\xi)| \le c_2 A \left(\beta_{m+1}^k |L_{m+1}(\xi)|\right)^{-\alpha_{m+1}}, \tag{2.6}$$

$$|\hat{\tau}_k^{(m)}(\xi)| \le c_3 A \left(\beta_{m+1}^{k+1} |L_{m+1}(\xi)|\right)^{\alpha_{m+1}}, \tag{2.7}$$

for all $k \in \mathbb{Z}$ satisfying $k \le L$ with some constants c_i $(1 \le i \le 3)$, where L is a negative integer, $L \le -4$, which will be determined in Lemma 2.4 below.

To prove Lemma 2.3 we need the following two lemmas.

Lemma 2.4 *Let $1 < q \le 2$, $\Omega \in L^q(S^{n-1})$, $h \in \Lambda_1^\eta$ and let σ_k be as in (2.1). Then, there exist a positive integer M, a positive number $\epsilon_0 \in (0, 1/4)$ and a negative integer L, $L \le -4$, such that*

$$|\hat{\sigma}_k(\xi)| \le C(\log \rho) \left(|\xi|\rho^{kM}\right)^{-\epsilon_0/q'} \|h\|_{\Lambda_1^\eta} \|\Omega\|_q$$

for $k \le L$. The constants M, ϵ_0, L and C are independent of ρ, q, h and Ω.

Lemma 2.5 *Let $\rho \ge 2$, $k \in \mathbb{Z}$, $1 < q \le 2$, $h \in \Lambda_1^\eta$ and $\Omega \in L^q(S^{n-1})$. Let P be a real-valued polynomial on $\mathbb{R}^n$ of degree $m \ge 1$. Write*

$$P(x) = \sum_{|\alpha|=m} a_\alpha y^\alpha + Q(y),$$

where $\deg Q \le m-1$ *if* $Q \ne 0$. *Then there exists a constant* $C > 0$ *independent of* ρ, k, q, h, Ω *and the coefficients of the polynomial* P *such that*

$$\left|\int_{\rho^k \le |y| < \rho^{k+1}} \exp\left(iP(x)\right) h(|x|)\Omega(x')|x|^{-n}\,dx\right| \le C(\log\rho)\|h\|_{\Lambda_1^\eta}\|\Omega\|_q \left(\rho^{km}\sum_{|\alpha|=m}|a_\alpha|\right)^{-\tau/(mq')},$$

where $\tau = 4^{-1}\min(1,\eta)$.

We can prove Lemma 2.5 similarly to the proof of Lemma 2.4 of [4]. To prove Lemma 2.4 we need the following two results, which can be found in [4].

Lemma 2.6 *Let* $\Phi : B(0,1) \to \mathbb{R}^d$ *be smooth and of finite type at the origin. Define* $G_m : B(0,1)\times S^{d-1} \to \mathbb{R}$ *by*

$$G_m(x,\xi) = \sum_{|\alpha|=m} \langle \xi, \partial_x^\alpha \Phi(x)\rangle x^\alpha \frac{m!}{\alpha!}$$

for $m \ge 1$. *Then, there exist constants* $R, \delta \in (0, 1/4)$ *and a mapping* ℓ *from* S^{d-1} *to a finite set of positive integers such that*

$$C_\Phi := \sup_{\xi\in S^{d-1}} \int_{|x|\le R} |G_{\ell(\xi)}(x,\xi)|^{-\delta}\,dx < \infty.$$

Lemma 2.7 *Let* $\psi, \varphi \in C^\infty(\mathbb{R})$ *be real-valued. Let* $s \in (0,1]$ *and* $a, b \in \mathbb{R}$ *with* $a < b$. *Suppose that* φ *is compactly supported and that*

$$|(d/dx)^k\psi(x)| \le s \quad \text{for } x \in [a,b],$$
$$|(d/dx)^{(k+1)}\psi(x)| \le 1 \quad \text{for } x \in [a-s, b+s],$$

where k *is a positive integer. Then, there exists a positive constant* C *depending only on* k *and* φ *such that*

$$\left|\int_a^b \exp(i\lambda\psi(x))\varphi(x)\,dx\right| \le C|\lambda|^{-\epsilon/k}\int_{a-s}^{b+s} |(d/dx)^k\psi(x)|^{-\epsilon(1+1/k)}\,dx$$

for all $\lambda \in \mathbb{R}\setminus\{0\}$ *and* $\epsilon \in (0,1]$.

Define a function F on an appropriate subinterval of $\mathbb{R}_+$ by $F(t) = \langle \xi, \Phi(tx)\rangle$ for fixed $\xi \in S^{d-1}$ and $x \in B(0,1)$. Then, we note that $(d/dt)^m F(t) = t^{-m}G_m(tx,\xi)$, where G_m is as in Lemma 2.6.

Proof of Lemma 2.4: Take an integer $\nu \ge 1$ and $a \in [2,4]$ such that $\rho = a^\nu$. Let Φ, δ, R and $\ell(\xi)$ be as in Lemma 2.6. Put $\ell_0 = \max_{\xi\in S^{d-1}} \ell(\xi)$. Let L be a negative integer such that

$$\left|(d/dr)^\ell \langle \xi', \Phi(\rho^k sr\theta)\rangle\right| < 1/2$$

for $1 \le \ell \le \ell_0 + 1$, $s \in [1, \rho]$, $r \in (0, 5)$, $\xi' \in S^{d-1}$ and $\theta \in S^{n-1}$ whenever $k \le L$ and such that $2^{k+2} < R$ if $k \le L$. Then, when $\xi \in \mathbb{R}^d \setminus \{0\}$ and $k \le L$, we write

$$\begin{aligned}\hat{\sigma}_k(\xi) &= \sum_{j=0}^{\nu-1} \int_{\rho^k a^j}^{\rho^k a^{j+1}} \int_{S^{n-1}} \exp\left(-2\pi i \langle \xi, \Phi(r\theta)\rangle\right) h(r)\Omega(\theta)\, d\sigma(\theta)\, dr/r \\ &= \sum_{j=0}^{\nu-1} \int_1^a \int_{S^{n-1}} \exp\left(-2\pi i \langle \xi, \Phi(\rho^k a^j r\theta)\rangle\right) h(\rho^k a^j r)\Omega(\theta)\, d\sigma(\theta)\, dr/r.\end{aligned}$$

Let $\phi \in C^\infty(\mathbb{R})$ satisfy $\mathrm{supp}(\phi) \subset (0, 10^{-9})$, $\phi \ge 0$, $\int \phi(s)\, ds = 1$. Define $h_j(r) = \int_{s<r/2} h(\rho^k a^j (r-s))\phi_u(s)\, ds$, $r > 0$, where $\phi_u(s) = u^{-1}\phi(u^{-1}s)$, $u > 0$. Then, if $u < 1$,

$$\int_1^a |h(\rho^k a^j r) - h_j(r)|\, dr/r \le C\omega(h, u). \tag{2.8}$$

We take $u = (|\xi|\rho^{kM})^{-\zeta/q'}$ for a suitable M with $M \ge \ell_0$ and $\zeta > 0$, which will be specified below. We assume $|\xi|\rho^{kM} \ge 1$ for the moment. Define

$$s_k(\xi) = \sum_{j=0}^{\nu-1} \int_1^a \int_{S^{n-1}} \exp\left(-2\pi i \langle \xi, \Phi(\rho^k a^j r\theta)\rangle\right) h_j(r)\Omega(\theta)\, d\sigma(\theta)\, dr/r.$$

Then, by (2.8)

$$\begin{aligned}|\hat{\sigma}_k(\xi) - s_k(\xi)| &\le C(\log\rho)\|\Omega\|_1 \omega(h, u) \\ &\le C(\log\rho)\|\Omega\|_1 \|h\|_{\Lambda^\eta} (|\xi|\rho^{kM})^{-\eta\zeta/q'},\end{aligned} \tag{2.9}$$

where we have used the fact that $\nu \approx \log\rho$.

By Lemma 2.7

$$\begin{aligned}&\left|\int_1^w \exp\left(-2\pi i \langle \xi, \Phi(\rho^k a^j t\theta)\rangle\right)\, dt\right| \\ &\quad\le C|\xi|^{-\epsilon/\ell(\xi')} \int_{1/2}^{a+1/2} \left|G_{\ell(\xi')}(\rho^k a^j r\theta, \xi')\right|^{-\epsilon(1+1/\ell(\xi'))}\, dr\end{aligned}$$

for $w \in [1, a]$, where $\xi' = \xi/|\xi|$. Also, $|h_j(a)| \le Cu^{-1}\|h\|_{\Delta_1}$, $\int_1^a |h_j(r)|\, dr/r \le C\|h\|_{\Delta_1}$, $\int_1^a |h'_j(r)|\, dr/r \le Cu^{-1}\|h\|_{\Delta_1}$. Therefore, applying integration by parts, we see that

$$\begin{aligned}&\left|\int_1^a \exp\left(-2\pi i \langle \xi, \Phi(\rho^k a^j r\theta)\rangle\right) h_j(r)\, dr/r\right| \\ &\quad\le Cu^{-1}\|h\|_{\Delta_1} |\xi|^{-\epsilon/\ell(\xi')} \int_{1/2}^{a+1/2} \left|G_{\ell(\xi')}(\rho^k a^j r\theta, \xi')\right|^{-\epsilon(1+1/\ell(\xi'))}\, dr/r.\end{aligned}$$

Note that

$$\int_{S^{n-1}}\left(\int_{1/2}^{a+1/2}\left|G_{\ell(\xi')}(\rho^k a^j r\theta,\xi')\right|^{-\epsilon(1+1/\ell(\xi'))}\,dr/r\right)|\Omega(\theta)|\,d\sigma(\theta)$$
$$\le C(\rho^k a^j)^{-n}\int_{|x|\le 2\rho^k a^{j+1}}\left|G_{\ell(\xi')}(x,\xi')\right|^{-2\epsilon}|\Omega(x')|\,dx =: I,$$

where $\epsilon\in(0,1]$. Since $2\rho^k a^{j+1}<R$, by Hölder's inequality we have

$$I\le C(\rho^k a^j)^{-n}(\rho^k a^j)^{n/q}\|\Omega\|_q\left(\int_{|x|\le R}\left|G_{\ell(\xi')}(x,\xi')\right|^{-2\epsilon q'}\,dx\right)^{1/q'}.$$

Therefore

$$\sum_{j=0}^{\nu-1}(\rho^k a^j)^{-n}\int_{|x|\le 2\rho^k a^{j+1}}\left|G_{\ell(\xi')}(x,\xi')\right|^{-2\epsilon}|\Omega(x')|\,dx$$
$$\le C\|\Omega\|_q\rho^{-kn/q'}\left(\sum_{j=0}^{\nu-1}a^{-jn/q'}\right)\left(\int_{|x|\le R}\left|G_{\ell(\xi')}(x,\xi')\right|^{-2\epsilon q'}\,dx\right)^{1/q'}$$
$$\le C(\log\rho)\|\Omega\|_q\rho^{-kn/q'}\left(\int_{|x|\le R}\left|G_{\ell(\xi')}(x,\xi')\right|^{-2\epsilon q'}\,dx\right)^{1/q'},$$

since $\nu\approx\log\rho$. Using these estimates, we have

$$\left|\sum_{j=0}^{\nu-1}\int_1^a\int_{S^{n-1}}\exp\left(-2\pi i\langle\xi,\Phi(\rho^k a^j r\theta)\rangle\right)h_j(r)\Omega(\theta)\,d\sigma(\theta)\,dr/r\right|$$
$$\le C(\log\rho)u^{-1}\|h\|_{\Delta_1}|\xi|^{-\epsilon/\ell(\xi')}\|\Omega\|_q\rho^{-kn/q'}\left(\int_{|x|\le R}\left|G_{\ell(\xi')}(x,\xi')\right|^{-2\epsilon q'}\,dx\right)^{1/q'},$$

where C is independent of ϵ, ρ, q, h and Ω. If we put $\epsilon=\delta/(2q')$, then by Lemma 2.6 we have

$$|s_k(\xi)|\le CC_\Phi^{1/q'}(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_q(|\xi|\rho^{kM})^{\zeta/q'}(|\xi|\rho^{2kn\ell(\xi')/\delta})^{-\delta/(2q'\ell(\xi'))}.$$

Therefore, if M is a positive integer such that $M-1<2n\ell_0/\delta\le M$ and $\zeta<\delta/(2\ell_0)$,

$$|s_k(\xi)|\le CC_\Phi^{1/q'}(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_q(|\xi|\rho^{kM})^{-(\delta/(2\ell_0)-\zeta)/q'}. \tag{2.10}$$

Combining (2.9) and (2.10), we can see that

$$|\hat{\sigma}_k(\xi)|\le C(\log\rho)\|h\|_{\Lambda_1^\eta}\|\Omega\|_q(|\xi|\rho^{kM})^{-\epsilon_0/q'},$$

where $\epsilon_0=\min(\eta\zeta,\delta/(2\ell_0)-\zeta)$. If $|\xi|\rho^{kM}\le 1$, the conclusion of Lemma 2.4 follows from the estimate $|\hat{\sigma}_k(\xi)|\le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1$ (see (2.14) below with $m=\ell+1$). This completes the proof of Lemma 2.4. □

Proof of Lemma 2.5: Let

$$I(x) = \int_1^\rho \exp\left(i\left[(\rho^k t)^m \sum_{|\alpha|=m} a_\alpha x^\alpha + Q(\rho^k t x)\right]\right) h(\rho^k t)\, dt/t.$$

Note that

$$\int_{\rho^k \le |y| < \rho^{k+1}} \exp\left(iP(x)\right) h(|x|)\Omega(x')|x|^{-n}\, dx = \int_{S^{n-1}} \Omega(\theta) I(\theta)\, d\sigma(\theta).$$

Let $a \in [2,4]$ and $\nu \ge 1$ be as in the proof of Lemma 2.4. Decompose $I(x) = \sum_{j=0}^{\nu-1} I_j(x)$, where

$$I_j(x) = \int_1^a \exp\left(i\left[(\rho^k a^j t)^m \sum_{|\alpha|=m} a_\alpha x^\alpha + Q(\rho^k a^j t x)\right]\right) h(\rho^k a^j t)\, dt/t.$$

Let $h_j(t) = \int_{s<t/2} h(\rho^k a^j (t-s))\phi_u(s)\, ds$ be as in the proof of Lemma 2.4 and

$$\tilde{I}_j(x) = \int_1^a \exp\left(i\left[(\rho^k a^j t)^m \sum_{|\alpha|=m} a_\alpha x^\alpha + Q(\rho^k a^j t x)\right]\right) h_j(t)\, dt/t.$$

Then by (2.8) $|I_j(x) - \tilde{I}_j(x)| \le C\omega(h,u)$, $0 < u < 1$. So,

$$\begin{aligned} &\left|\int_{S^{n-1}} \Omega(\theta) I_j(\theta)\, d\sigma(\theta) - \int_{S^{n-1}} \Omega(\theta)\tilde{I}_j(\theta)\, d\sigma(\theta)\right| \\ &\quad \le \int_{S^{n-1}} |\Omega(\theta)||I_j(\theta) - \tilde{I}_j(\theta)|\, d\sigma(\theta) \\ &\quad \le C\omega(h,u)\|\Omega\|_1 \le C\|h\|_{\Lambda^\eta}\|\Omega\|_1 u^\eta \end{aligned} \tag{2.11}$$

for $0 \le j \le \nu - 1$. Also, since $|I(x)| \le C(\log\rho)\|h\|_{\Delta_1}$,

$$\left|\int_{S^{n-1}} \Omega(\theta) I(\theta)\, d\sigma(\theta)\right| \le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1. \tag{2.12}$$

Now, we assume that $b := \rho^{km}\sum_{|\alpha|=m} |a_\alpha| \ge 1$ and put $u = (a^{jm}b)^{-1/(4mq')}$. Then, as in the proof of Lemma 2.4, an integration by parts argument implies that

$$|\tilde{I}_j(x)| \le Cu^{-1}\|h\|_{\Delta_1}\left|(\rho^k a^j)^m \sum_{|\alpha|=m} a_\alpha x^\alpha\right|^{-1/m}, \tag{2.13}$$

since

$$\begin{aligned} &\left|\int_1^w \exp\left(i\left[(\rho^k a^j t)^m \sum_{|\alpha|=m} a_\alpha x^\alpha + Q(\rho^k a^j t x)\right]\right) dt\right| \\ &\quad \le C\left|(\rho^k a^j)^m \sum_{|\alpha|=m} a_\alpha x^\alpha\right|^{-1/m} \end{aligned}$$

for $w \in [1, a]$, which follows from van der Corput's lemma. We also have $|\tilde{I}_j(x)| \le C\|h\|_{\Delta_1}$. Combining this with (2.13), we have

$$|\tilde{I}_j(x)| \le Cu^{-1}\|h\|_{\Delta_1} \min\left(1, \left|(\rho^k a^j)^m \sum_{|\alpha|=m} a_\alpha x^\alpha\right|^{-1/(2mq')}\right)$$

and hence by Hölder's inequality and [7, Corollary 1]

$$\begin{aligned}\left|\int_{S^{n-1}} \Omega(\theta)\tilde{I}_j(\theta)\, d\sigma(\theta)\right| &\le \int_{S^{n-1}} |\Omega(\theta)\tilde{I}_j(\theta)|\, d\sigma(\theta) \le \|\Omega\|_q \|\tilde{I}_j\|_{q'} \\ &\le Cu^{-1}\|h\|_{\Delta_1}\|\Omega\|_q \left(\int_{S^{n-1}} \left|(\rho^k a^j)^m \sum_{|\alpha|=m} a_\alpha \theta^\alpha\right|^{-1/(2m)} d\sigma(\theta)\right)^{1/q'} \\ &\le C\|h\|_{\Delta_1}\|\Omega\|_q \left((\rho^k a^j)^m \sum_{|\alpha|=m} |a_\alpha|\right)^{-1/(4mq')}.\end{aligned}$$

By this estimate and (2.11) we see that

$$\begin{aligned}&\left|\int_{S^{n-1}} \Omega(\theta) I_j(\theta)\, d\sigma(\theta)\right| \\ &\quad \le C\left(\|h\|_{\Lambda^\eta}\|\Omega\|_1 + \|h\|_{\Delta_1}\|\Omega\|_q\right)\left((\rho^k a^j)^m \sum_{|\alpha|=m} |a_\alpha|\right)^{-\tau/(mq')},\end{aligned}$$

where $\tau = 4^{-1}\min(1, \eta)$. Thus

$$\begin{aligned}\left|\int_{S^{n-1}} \Omega(\theta) I(\theta)\, d\sigma(\theta)\right| &\le \sum_{j=0}^{\nu-1}\left|\int_{S^{n-1}} \Omega(\theta) I_j(\theta)\, d\sigma(\theta)\right| \\ &\le C(\log\rho)\left(\|h\|_{\Lambda^\eta}\|\Omega\|_1 + \|h\|_{\Delta_1}\|\Omega\|_q\right)\left(\rho^{km}\sum_{|\alpha|=m}|a_\alpha|\right)^{-\tau/(mq')},\end{aligned}$$

if $\rho^{km}\sum_{|\alpha|=m}|a_\alpha| \ge 1$. Along with (2.12), this implies the conclusion of Lemma 2.5. □

Proof of Lemma 2.3: We easily see that

$$\|\sigma_k^{(m)}\| \le C\|\Omega\|_1 \int_{\rho^k}^{\rho^{k+1}} |h(r)|\, dr/r \le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1 \tag{2.14}$$

for $1 \le m \le \ell + 1$. By (2.14) and (2.2) we have

$$\|\tau_k^{(m)}\| \le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1 \tag{2.15}$$

for $1 \le m \le \ell$. By (2.15) and Hölder's inequality we have (2.5).

Let $k \le L$, where L is as in Lemma 2.4. By Lemmas 2.4 and 2.5 we have $|\hat{\sigma}_k^{(m)}(\xi)| \le CA\left(\beta_m^k|L_m(\xi)|\right)^{-\alpha_m}$ for $m = 2, \dots, \ell+1$. Also, we note that $|\Phi_{k,m}(\xi)|$ is bounded by $C\left(\beta_{m+1}^k|L_{m+1}(\xi)|\right)^{-N}$ for all $N > 0$, when $1 \le m \le \ell$. Using these estimates and (2.14) in the definition of $\tau_k^{(m)}$ in (2.2), we have (2.6).

To prove (2.7), we note that

$$\left|\hat{\sigma}_k^{(m+1)}(\xi) - \hat{\sigma}_k^{(m)}(\xi)\right| \le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1\beta_{m+1}^{k+1}|L_{m+1}(\xi)|. \qquad (2.16)$$

Also, by (2.3) we see that

$$\left|\Phi_{k,m+1}(\xi) - \Phi_{k,m}(\xi)\right| \le C\beta_{m+1}^k|L_{m+1}(\xi)|. \qquad (2.17)$$

The estimates (2.14), (2.16) and (2.17) imply

$$|\hat{\tau}_k^{(m)}(\xi)| \le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1\beta_{m+1}^{k+1}|L_{m+1}(\xi)|, \qquad (2.18)$$

since

$$|\hat{\tau}_k^{(m)}(\xi)| \le \left|\left(\hat{\sigma}_k^{(m+1)}(\xi) - \hat{\sigma}_k^{(m)}(\xi)\right)\Phi_{k,m+1}(\xi)\right| + \left|\left(\Phi_{k,m+1}(\xi) - \Phi_{k,m}(\xi)\right)\hat{\sigma}_k^{(m)}(\xi)\right|.$$

By (2.15) we also have $|\hat{\tau}_k^{(m)}(\xi)| \le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1$. This estimate and (2.18) imply (2.7). This completes the proof of Lemma 2.3. □

Proof of Proposition 2.2: Let $\tilde{T}_\rho^{(m)}(f) = \sum_{k\le L}\tau_k^{(m)} * f$ for $1 \le m \le \ell$, where L is as in Lemma 2.3. Then, to prove Proposition 2.2 it suffices to show a version of Proposition 2.2 for $\tilde{T}_\rho^{(m)}$ with bounds similar to those for $T_\rho^{(m)}$, since $\|T_\rho^{(m)}(f) - \tilde{T}_\rho^{(m)}(f)\|_p \le CA\|f\|_p$ for $1 \le p \le \infty$, where A is as in Lemma 2.3. Let $\{\psi_k\}_{-\infty}^{\infty}$ be a sequence of non-negative functions in $C^\infty(\mathbb{R})$ such that each ψ_k is supported in $[\beta_{m+1}^{-k-1}, \beta_{m+1}^{-k+1}]$, $\sum_k\psi_k(t)^2 = 1$ for $t > 0$ and

$$|(d/dt)^j\psi_k(t)| \le c_j|t|^{-j}, \quad j = 1, 2, \dots,$$

where the constants c_j are independent of β_{m+1}. This is possible since $\beta_{m+1} \ge 2$. Let

$$\left(S_k^{(m+1)}(f)\right)\hat{}\,(\xi) = \psi_k\left(|H_{m+1}\pi_{s_{m+1}}^d R_{m+1}(\xi)|\right)\hat{f}(\xi).$$

We also write $S_k^{(m+1)} = S_k$. Put

$$D_j^{(m)}(f) = \sum_{k=-\infty}^{L} S_{j+k}\left(\tau_k^{(m)} * S_{j+k}(f)\right).$$

Then $\tilde{T}_\rho^{(m)} = \sum_j D_j^{(m)}$. Plancherel's theorem and the estimates (2.5)–(2.7) imply that

$$\begin{aligned}
\left\|D_j^{(m)}(f)\right\|_2^2 &\le \sum_{k\le L} C\int_{\Delta(j+k)}|\hat{\tau}_k^{(m)}(\xi)|^2|\hat{f}(\xi)|^2\,d\xi \\
&\le CA^2\min\left(1, \beta_{m+1}^{-2\alpha_{m+1}(|j|-2)}\right)\sum_{k\le L}\int_{\Delta(j+k)}|\hat{f}(\xi)|^2\,d\xi \\
&\le CA^2\min\left(1, \beta_{m+1}^{-2\alpha_{m+1}(|j|-2)}\right)\|f\|_2^2,
\end{aligned}$$

where $\Delta(k) = \{\beta_{m+1}^{-k-1} \leq |H_{m+1}\pi_{s_{m+1}}^d R_{m+1}(\xi)| \leq \beta_{m+1}^{-k+1}\}$. Thus we have

$$\left\| D_j^{(m)}(f) \right\|_2 \leq CA \min\left(1, \beta_{m+1}^{-\alpha_{m+1}(|j|-2)}\right) \|f\|_2. \tag{2.19}$$

By (2.19) we have

$$\|\tilde{T}_\rho^{(m)}(f)\|_2 \leq \sum_j \|D_j^{(m)}(f)\|_2 \leq CAB\|f\|_2, \tag{2.20}$$

since $B \geq \left(1 - \beta_{m+1}^{-\alpha_{m+1}}\right)^{-1}$, where B is as in Proposition 2.1.

Taking Proposition 2.1 for granted for the moment and recalling the definition of $\tau_k^{(m)}$ in (2.2), by change of variables and a well-known theorem for L^p boundedness of maximal functions (see [5, Section 6]) we have

$$\begin{aligned} \left\| (\tau^{(m)})^*(f) \right\|_p &\leq C \left\| (\mu^{(m+1)})^*(|f|) \right\|_p + C \left\| (\mu^{(m)})^*(|f|) \right\|_p \\ &\leq C_p AB^{2/p} \|f\|_p \end{aligned} \tag{2.21}$$

for $p > 1 + \theta$.

By (2.5), (2.21) and the proof of Lemma in [3, p. 544], we have the following.

Lemma 2.8 *Let* $u \in (1+\theta, 2]$, $1/v - 1/2 = 1/(2u)$. *Then we have*

$$\left\| \left(\sum_{k \leq L} |\tau_k^{(m)} * g_k|^2 \right)^{1/2} \right\|_v \leq (c_1 C_u)^{1/2} AB^{1/u} \left\| \left(\sum_{k \leq L} |g_k|^2 \right)^{1/2} \right\|_v,$$

where the constants c_1 *and* C_u *are as in* (2.5) *and* (2.21), *respectively.*

Also, the Littlewood–Paley theory implies that

$$\|D_j^{(m)}(f)\|_p \leq c_p \left\| \left(\sum_{k \leq L} |\tau_k^{(m)} * S_{j+k}(f)|^2 \right)^{1/2} \right\|_p, \tag{2.22}$$

$$\left\| \left(\sum_k |S_k(f)|^2 \right)^{1/2} \right\|_p \leq c_p \|f\|_p, \tag{2.23}$$

where $1 < p < \infty$ and c_p is independent of β_{m+1} and the linear transformations R_{m+1}, H_{m+1}.

Let $1 + \theta < p \leq 4/(3-\theta)$. Then, there exists $u \in (1+\theta, 2]$ such that $1/p = 1/2 + (1-\theta)/(2u)$. Let $1/v - 1/2 = 1/(2u)$. Then, by (2.22), (2.23) and Lemma 2.8 we have

$$\|D_j^{(m)}(f)\|_v \leq CAB^{1/u} \|f\|_v, \tag{2.24}$$

where C is independent of ρ and the linear transformations R_i, H_i, $2 \le i \le \ell+1$. Noting that $1/p = \theta/2 + (1-\theta)/v$ and interpolating between (2.19) and (2.24), we have

$$\|D_j^{(m)}(f)\|_p \le CAB^{(1-\theta)/u} \min\left(1, \beta_{m+1}^{-\theta\alpha_{m+1}(|j|-2)}\right) \|f\|_p,$$

which implies that

$$\begin{aligned} \|\tilde{T}_\rho^{(m)}(f)\|_p &\le \sum_j \|D_j^{(m)}(f)\|_p \le CAB^{(1-\theta)/u} \left(1-\beta_{m+1}^{-\theta\alpha_{m+1}}\right)^{-1} \|f\|_p \\ &\le CAB^{2/p}\|f\|_p. \end{aligned} \tag{2.25}$$

A duality and interpolation argument using (2.20) and (2.25) implies the conclusion of Proposition 2.2 with $T_\rho^{(m)}$ replaced by $\tilde{T}_\rho^{(m)}$, which proves Proposition 2.2. □

We now prove Proposition 2.1 by induction on j. First, the inequality $(\mu^{(1)})^*(f)(x) \le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1|f(x-P(0))|$ implies the estimate (2.4) for $j=1$. Next, we prove (2.4) for $j=m$ by assuming (2.4) for $j=m-1$, $2 \le m \le \ell+1$. Define a sequence $\eta^{(m)} = \{\eta_k^{(m)}\}_{k=-\infty}^{-1}$ of Borel measures on $\mathbb{R}^d$ by

$$\hat{\eta}_k^{(m)}(\xi) = \varphi\left(\beta_m^k|H_m\pi_{s_m}^d R_m(\xi)|\right)\hat{\mu}_k^{(m-1)}(\xi),$$

where $\varphi \in C_0^\infty(\mathbb{R})$ is as in the definition of $\tau_k^{(m)}$ in (2.2). Then, from (2.4) with $j=m-1$, it follows that

$$\left\|(\eta^{(m)})^*(f)\right\|_p \le C\left\|(\mu^{(m-1)})^*(f)\right\|_p \le CAB^{2/p}\|f\|_p \tag{2.26}$$

for $p > 1+\theta$, where A, B are as above. As in the proof of Lemma 2.3, we have

$$\begin{aligned} \|\eta_k^{(m)}\| + \|\mu_k^{(m)}\| &\le C\|\mu_k^{(m-1)}\| + \|\mu_k^{(m)}\| \\ &\le C\|\Omega\|_1 \int_{\rho^k}^{\rho^{k+1}} |h(r)|\,dr/r \\ &\le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1 \le CA. \end{aligned} \tag{2.27}$$

Let $k \le L$, where L is as above. Since

$$\begin{aligned} &|\hat{\mu}_k^{(m)}(\xi) - \hat{\eta}_k^{(m)}(\xi)| \\ &\quad\le |\hat{\mu}_k^{(m)}(\xi) - \hat{\mu}_k^{(m-1)}(\xi)| + \left|\left(\varphi\left(\beta_m^k|H_m\pi_{s_m}^d R_m(\xi)|\right)-1\right)\hat{\mu}_k^{(m-1)}(\xi)\right|, \end{aligned}$$

arguing as in the proof of (2.7), we see that

$$\begin{aligned} |\hat{\mu}_k^{(m)}(\xi) - \hat{\eta}_k^{(m)}(\xi)| &\le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1\left(\beta_m^{k+1}|L_m(\xi)|\right)^{\alpha_m} \\ &\le CA\left(\beta_m^{k+1}|L_m(\xi)|\right)^{\alpha_m}. \end{aligned} \tag{2.28}$$

We also have the following:

$$|\hat{\mu}_k^{(m)}(\xi)| \le CA\left(\beta_m^k |L_m(\xi)|\right)^{-\alpha_m}, \tag{2.29}$$

$$\begin{aligned} |\hat{\eta}_k^{(m)}(\xi)| &\le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1\left(\beta_m^k |L_m(\xi)|\right)^{-\alpha_m} \\ &\le CA\left(\beta_m^k |L_m(\xi)|\right)^{-\alpha_m}. \end{aligned} \tag{2.30}$$

We can prove the estimate (2.29) arguing as in the proof of (2.6). The definition of $\eta_k^{(m)}$ and (2.27) imply the first inequality of (2.30).

We have only to prove (2.4) with $j = m$ for $p \in (1+\theta, 2]$, since the estimate (2.4) for $p > 2$ follows from interpolation between the estimate (2.4) for $p \in (1+\theta, 2]$ and the obvious estimate $\|(\mu^{(m)})^*(f)\|_\infty \le CA\|f\|_\infty$. Let

$$g_m(f)(x) = \left(\sum_{k\le L}\left|\nu_k^{(m)} * f(x)\right|^2\right)^{1/2},$$

where $\nu_k^{(m)} = \mu_k^{(m)} - \eta_k^{(m)}$. Then, we see that

$$(\tilde{\mu}^{(m)})^*(f) \le g_m(f) + (\eta^{(m)})^*(|f|), \tag{2.31}$$

where $(\tilde{\mu}^{(m)})^*(f) = \sup_{k\le L}|\mu_k^{(m)} * f|$. Note that to prove (2.4) with $j = m$ it suffices to prove it with $(\tilde{\mu}^{(m)})^*$ in place of $(\mu^{(m)})^*$. Since we have (2.26) and (2.31), to show (2.4) with $j = m$ it suffices to prove $\|g_m(f)\|_p \le CAB^{2/p}\|f\|_p$ for $p \in (1+\theta, 2]$. Let

$$U_\epsilon^{(m)}(f) = \sum_{k\le L}\epsilon_k \nu_k^{(m)} * f,$$

where $\epsilon = \{\epsilon_k\}$, $\epsilon_k = 1$ or -1. Then, we shall show that

$$\left\|U_\epsilon^{(m)}(f)\right\|_p \le CAB^{2/p}\|f\|_p \tag{2.32}$$

for $p \in (1+\theta, 2]$, where C is independent of ϵ. The desired estimate follows from (2.32) by a well-known property of Rademacher's functions.

To prove (2.32) we use the following:

Lemma 2.9 *Let $\{p_j\}_1^\infty$ be a sequence of real numbers defined by $p_1 = 2$ and $1/p_{j+1} = 1/2 + (1-\theta)/(2p_j)$ for $j \ge 1$. Then, we have*

$$\left\|U_\epsilon^{(m)}(f)\right\|_{p_j} \le C_j AB^{2/p_j}\|f\|_{p_j} \quad \textit{for } j \ge 1.$$

We can see that $1/p_j = (1-a^j)/(1+\theta)$, where $a = (1-\theta)/2$. Thus $\{p_j\}$ is decreasing and converges to $1+\theta$. We can prove Lemma 2.9 by (2.26)–(2.30).

Proof: Define

$$U_j^{(m)}(f) = \sum_{k=-\infty}^{L} \epsilon_k S_{j+k}\left(\nu_k^{(m)} * S_{j+k}(f)\right),$$

where $S_k = S_k^{(m)}$ (the operators $S_k^{(m)}$ are as in the proof of Proposition 2.2). Then, $U_\epsilon^{(m)} = \sum_j U_j^{(m)}$. Arguing as in the proof of (2.19), and using Plancherel's theorem and the estimates (2.27)–(2.30), we have

$$\left\|U_j^{(m)}(f)\right\|_2 \le CA \min\left(1, \beta_m^{-\alpha_m(|j|-2)}\right) \|f\|_2, \tag{2.33}$$

and hence $\left\|U_\epsilon^{(m)}(f)\right\|_2 \le \sum_j \|U_j^{(m)}(f)\|_2 \le CAB\|f\|_2$. This proves the assertion of Lemma 2.9 for $j = 1$.

We now assume the estimate of Lemma 2.9 for $j = s$ and prove it for $j = s + 1$. By induction, this will complete the proof of Lemma 2.9. From the estimate (2.31), it follows that

$$(\tilde{\nu}^{(m)})^*(f) \le (\tilde{\mu}^{(m)})^*(|f|) + (\eta^{(m)})^*(|f|) \le g_m(|f|) + 2(\eta^{(m)})^*(|f|),$$

where $(\tilde{\nu}^{(m)})^*(f) = \sup_{k\le L} ||\nu_k^{(m)}| * f|$. By our assumption we have $\|g_m(f)\|_{p_s} \le CAB^{2/p_s}\|f\|_{p_s}$. This estimate and (2.26) imply

$$\begin{aligned}\left\|(\tilde{\nu}^{(m)})^*(f)\right\|_{p_s} &\le \|g_m(|f|)\|_{p_s} + 2\left\|(\eta^{(m)})^*(|f|)\right\|_{p_s} \\ &\le CAB^{2/p_s}\|f\|_{p_s}.\end{aligned} \tag{2.34}$$

Arguing as in the proof of (2.25), and using (2.27), (2.33) and (2.34), we can now obtain the estimate of Lemma 2.9 for $j = s + 1$. This completes the proof of Lemma 2.9. □

Let $p \in (1 + \theta, 2]$ and let $\{p_j\}_1^\infty$ be as in Lemma 2.9. Then, we can find a positive integer N such that $p_{N+1} < p \le p_N$. The estimate (2.32) now follows from interpolation between the estimates of Lemma 2.9 for $j = N$ and $j = N + 1$. This finishes the proof of (2.4) for $j = m$. By induction, this completes the proof of Proposition 2.1.

Proof of Theorem 1.1: By taking $\rho = 2^{q'}$ in Proposition 2.2 we see that

$$\|T_{2^{q'}}^{(m)}(f)\|_p \le C_\theta (q-1)^{-1}\|h\|_{\Lambda_1^\eta}\|\Omega\|_q\|f\|_p$$

for $p \in (1+\theta, (1+\theta)/\theta)$. This completes the proof of Theorem 1.1, since $T = \sum_{m=1}^{\ell} T_\rho^{(m)}$ and $(1 + \theta, (1 + \theta)/\theta) \to (1, \infty)$ as $\theta \to 0$. □

3 Estimates for maximal functions

Let

$$T^*(f)(x) = \sup_{\epsilon\in(0,1)} \left|\int_{\epsilon<|y|<1} f(x - \Phi(y))K(y)\,dy\right|, \tag{3.1}$$

where K is as in (1.2). Then, we have an analog of Theorem 1.1 for the maximal operator T^*.

Theorem 3.1 *Let $\Omega \in L^q(S^{n-1})$, $q \in (1, 2]$ and $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that Ω satisfies* (1.1). *Then*

$$\|T^*(f)\|_{L^p(\mathbb{R}^d)} \le C_p(q-1)^{-1}\|h\|_{\Lambda_1^\eta}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where C_p is independent of q, h and Ω.

By Theorem 3.1 and extrapolation we have the following result.

Theorem 3.2 *Let $\Omega \in L\log L(S^{n-1})$ and $h \in \Lambda_1^\eta$ for some $\eta > 0$. Suppose that Ω satisfies the condition* (1.1). *Let $T^* f$ be defined as in* (3.1) *with the functions h and Ω. Then*

$$\|T^*(f)\|_{L^p(\mathbb{R}^d)} \le C_p\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$.

If the function h is identically 1, then Theorem 3.2 was shown in [1].

To prove Theorem 3.1, we use the following result.

Lemma 3.3 *Let $\theta \in (0, 1)$ and let positive numbers $A = (\log\rho)\|h\|_{\Lambda_1^\eta}\|\Omega\|_q$, $B = \left(1-\rho^{-\theta\kappa/q'}\right)^{-1}$ be as above. Define*

$$T^*_{m,\rho}(f)(x) = \sup_{k\le L}\left|\sum_{j=k}^{L}\tau_j^{(m)} * f(x)\right| \tag{3.2}$$

for $1 \le m \le \ell$, where the measures $\tau_k^{(m)}$ are as in (2.2) *and L is as in Lemma* 1. *Let $I_\theta = (2(1+\theta)/(\theta^2-\theta+2), (1+\theta)/\theta)$. Then, we have*

$$\|T^*_{m,\rho}(f)\|_p \le CA\left(B^{1+\delta(p)} + B^{2/p+1-\theta/2}\right)\|f\|_p$$

for $p \in I_\theta$, where C is independent of $q \in (1, 2]$, $\Omega \in L^q(S^{n-1})$, $h \in \Lambda_1^\eta$ and ρ.

This can be proved by results in Section 2.

Proof: Let $\tilde{T}_\rho^{(m)}(f) = \sum_{k\le L}\tau_k^{(m)} * f$ be as in the proof of Proposition 2.2. Let φ_k be defined by

$$\hat{\varphi}_k(\xi) = \varphi\left(\beta_{m+1}^k|H_{m+1}\pi_{s_{m+1}}^d R_{m+1}(\xi)|\right),$$

where φ is as in the definition of $\tau_k^{(m)}$ in (2.2). We now decompose

$$\sum_{j=k}^{L}\tau_j^{(m)} * f = \varphi_k * \tilde{T}_\rho^{(m)}(f) - \varphi_k * \left(\sum_{j=-\infty}^{k-1}\tau_j^{(m)} * f\right) + (\delta-\varphi_k) * \left(\sum_{j=k}^{L}\tau_j^{(m)} * f\right),$$

where $k \le L$ and $\delta = \delta_0$ is the delta function on $\mathbb{R}^d$ (see [3, 5]). Then, we have

$$T^*_{\rho,m}(f) \le \sup_{k\le L} \left|\varphi_k * \tilde{T}^{(m)}_\rho(f)\right| + \sum_{j=0}^{\infty} M^{(m)}_j(f), \tag{3.3}$$

where

$$M^{(m)}_j(f) = \sup_{k\le L} \left|\varphi_k * \left(\tau^{(m)}_{k-j-1} * f\right)\right| + \sup_{k\le L-j} \left|(\delta - \varphi_k) * \left(\tau^{(m)}_{j+k} * f\right)\right|.$$

From Proposition 2.2 it follows that

$$\left\|\sup_{k\le L} \left|\varphi_k * \tilde{T}^{(m)}_\rho(f)\right|\right\|_p \le CAB^{1+\delta(p)}\|f\|_p \tag{3.4}$$

for $p \in (1+\theta, (1+\theta)/\theta)$, and the estimate (2.21) implies that

$$\|M^{(m)}_j(f)\|_r \le CAB^{2/r}\|f\|_r \quad \text{for } r > 1+\theta. \tag{3.5}$$

Since

$$M^{(m)}_j(f) \le \left(\sum_{k\le L-j} \left|(\delta-\varphi_k) * \left(\tau^{(m)}_{j+k} * f\right)\right|^2\right)^{1/2} + \left(\sum_{k\le L} \left|\varphi_k * \left(\tau^{(m)}_{k-j-1} * f\right)\right|^2\right)^{1/2},$$

arguing as in [5, p. 820] and using the estimates (2.6) and (2.7) along with Plancherel's theorem, we have

$$\|M^{(m)}_j(f)\|_2 \le CA\beta_{m+1}^{-\alpha_{m+1}j}\left(1-\beta_{m+1}^{-2\alpha_{m+1}}\right)^{-1/2}\|f\|_2. \tag{3.6}$$

We note that for any $p \in I_\theta$ there exists a number $r \in (1+\theta, 2(1+\theta)/\theta)$ such that $1/p = (1-\theta)/r + \theta/2$. Therefore, interpolating between (3.5) and (3.6), we have

$$\|M^{(m)}_j(f)\|_p \le CAB^{2(1-\theta)/r}\left(1-\beta_{m+1}^{-2\alpha_{m+1}}\right)^{-\theta/2}\beta_{m+1}^{-\alpha_{m+1}\theta j}\|f\|_p. \tag{3.7}$$

From (3.3), (3.4) and (3.7), it follows that

$$\|T^*_{\rho,m}(f)\|_p \le CA\left(B^{1+\delta(p)} + B^{2(1-\theta)/r+1}\left(1-\beta_{m+1}^{-2\alpha_{m+1}}\right)^{-\theta/2}\right)\|f\|_p$$

for $p \in I_\theta$. Using $\left(1-\beta_{m+1}^{-2\alpha_{m+1}}\right)^{-1} \le B$ and $2(1-\theta)/r + \theta/2 + 1 = 2/p + 1 - \theta/2$ in this estimate, we can obtain the conclusion of Lemma 3.3. □

Proof of Theorem 3.1: Let

$$T^*_\rho(f)(x) = \sup_{\epsilon\in(0,\rho^{L+1})} \left|\int_{\epsilon<|y|<\rho^{L+1}} f(x-\Phi(y))K(y)\,dy\right|.$$

Then, we have

$$T^*(f)(x) \le T^*_\rho(f)(x) + J_\rho(f)(x), \tag{3.8}$$

where $J_\rho(f)(x) = \int_{\rho^{L+1}\le|y|<1} |f(x-\Phi(y))||K(y)|\,dy$. We note that

$$T^*_\rho(f) \le T^*_{0,\rho}(f) + \mu^*_\rho(|f|), \tag{3.9}$$

where $\mu^*_\rho = (\mu^{(\ell+1)})^*$ is as in Proposition 2.1 and $T^*_{0,\rho}(f)$ is defined by the formula in (3.2) with $\{\tau_j^{(m)}\}_{j\le L}$ replaced by the sequence $\{\sigma_j\}_{j\le L}$ of measures in (2.1). Since $T^*_{0,\rho}(f) \le \sum_{m=1}^{\ell} T^*_{m,\rho}(f)$, using Lemma 3.3 with $\rho = 2^{q'}$, we see that

$$\|T^*_{0,2^{q'}}(f)\|_p \le C_\theta(q-1)^{-1}\|h\|_{\Lambda_1^\eta}\|\Omega\|_q\|f\|_p \tag{3.10}$$

for $p \in I_\theta$. Also, by Proposition 2.1 with $\rho = 2^{q'}$ we have

$$\|\mu^*_{2^{q'}}(|f|)\|_p \le C_\theta(q-1)^{-1}\|h\|_{\Lambda_1^\eta}\|\Omega\|_q\|f\|_p \tag{3.11}$$

for $p \in I_\theta$. Note that

$$\int_{\rho^{L+1}\le|y|<1} |K(y)|\,dy \le C(\log\rho)\|h\|_{\Delta_1}\|\Omega\|_1.$$

Therefore, it is easy to see that

$$\|J_{2^{q'}}(f)\|_p \le C(q-1)^{-1}\|h\|_{\Lambda_1^\eta}\|\Omega\|_q\|f\|_p \tag{3.12}$$

for $p \in I_\theta$. Since $I_\theta \to (1,\infty)$ as $\theta \to 0$, by (3.8)–(3.12) we obtain the conclusion of Theorem 3.1. □

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