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Littlewood-Paley equivalence and homogeneous Fourier multipliers

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Abstract. We consider certain Littlewood-Paley operators and prove characterization of some function spaces in terms of those operators. When treating weighted Lebesgue spaces, a generalization to weighted spaces will be made for Hörmander's theorem on the invertibility of homogeneous Fourier multipliers. Also, applications to the theory of Sobolev spaces will be given.

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1. Introduction

Let ψ be a function in $L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \psi(x)\, dx = 0. \tag{1.1}$$

We consider the Littlewood-Paley function on $\mathbb{R}^n$ defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2\, \frac{dt}{t} \right)^{1/2}, \tag{1.2}$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. The following result of Benedek, Calderón and Panzone [2] on the L^p boundedness, $1 < p < \infty$, of g_ψ is well-known.

Theorem A. *We assume* (1.1) *for* ψ *and*

$$|\psi(x)| \leq C(1 + |x|)^{-n-\epsilon}, \tag{1.3}$$

$$\int_{\mathbb{R}^n} |\psi(x - y) - \psi(x)|\, dx \leq C|y|^\epsilon \tag{1.4}$$

for some positive constant ϵ. Then g_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$:

$$\|g_\psi(f)\|_p \le C_p \|f\|_p, \tag{1.5}$$

where

$$\|f\|_p = \|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.$$

By the Plancherel theorem, it follows that g_ψ is bounded on $L^2(\mathbb{R}^n)$ if and only if $m \in L^\infty(\mathbb{R}^n)$, where $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 \, dt/t$, which is a homogeneous function of degree 0. Here the Fourier transform is defined as

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i \langle x, \xi \rangle} \, dx, \quad \langle x, \xi \rangle = \sum_{k=1}^n x_k \xi_k.$$

Let

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}$$

be the Poisson kernel on the upper half space $\mathbb{R}^n \times (0, \infty)$ and $Q(x) = [(\partial/\partial t) P_t(x)]_{t=1}$. Then, we can see that the function Q satisfies the conditions (1.1), (1.3) and (1.4). Thus by Theorem A g_Q is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

Let $H(x) = \text{sgn}(x)\chi_{[-1,1]}(x) = \chi_{[0,1]}(x) - \chi_{[-1,0]}(x)$ on $\mathbb{R}$ (the Haar function), where χ_E denotes the characteristic function of a set E and $\text{sgn}(x)$ the signum function. Then $g_H(f)$ is the Marcinkiewicz integral

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $F(x) = \int_0^x f(y) \, dy$. Also, we can easily see that Theorem A implies that g_H is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.

Further, we can consider the generalized Marcinkiewicz integral $\mu_\alpha(f)$ ($\alpha > 0$) on $\mathbb{R}$ defined by

$$\mu_\alpha(f)(x) = \left(\int_0^\infty |S_t^\alpha(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$S_t^\alpha(f)(x) = \frac{\alpha}{t} \int_0^t \left(1 - \frac{u}{t} \right)^{\alpha-1} (f(x-u) - f(x+u)) \, du.$$

We observe that $\mu_\alpha(f) = g_{\varphi^{(\alpha)}}(f)$ with

$$\varphi^{(\alpha)}(x) = \alpha |1 - |x||^{\alpha-1} \text{sgn}(x) \chi_{(-1,1)}(x). \tag{1.6}$$

The square function μ_1 coincides with the ordinary Marcinkiewicz integral μ. When ψ is compactly supported, relevant sharp results for the L^p boundedness of g_ψ can be found in [6, 8, 20].

We can also consider Littlewood-Paley operators on the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < \infty$. We consider a dense subspace $\mathcal{S}_0(\mathbb{R}^n)$ of $H^p(\mathbb{R}^n)$ consisting of those functions f in $\mathcal{S}(\mathbb{R}^n)$ which satisfy $\hat{f} = 0$ near the origin,

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of rapidly decreasing smooth functions. Let $f \in \mathcal{S}_0(\mathbb{R})$. Then, if $2/(2\alpha + 1) < p < \infty$ and $\alpha > 0$, we have $\|\mu_\alpha(f)\|_p \simeq \|f\|_{H^p}$, which means

$$c_p\|f\|_{H^p} \leq \|\mu_\alpha(f)\|_p \leq C_p\|f\|_{H^p} \qquad (1.7)$$

with some positive constants c_p, C_p independent of f (see [27], [19]).

To state results about the reverse inequality of (1.5), we first recall a theorem of Hörmander [12]. Let $m \in L^\infty(\mathbb{R}^n)$ and define

$$T_m(f)(x) = \int_{\mathbb{R}^n} m(\xi)\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\,d\xi. \qquad (1.8)$$

We say that m is a Fourier multiplier for L^p and write $m \in M^p$ if there exists a constant $C > 0$ such that

$$\|T_m(f)\|_p \leq C\|f\|_p$$

for all $f \in L^2 \cap L^p$. Then the result of Hörmander [12] can be stated as follows.

Theorem B. *Let m be a bounded function on $\mathbb{R}^n$ which is homogeneous of degree 0. Suppose that $m \in M^p$ for all $p \in (1,\infty)$. Suppose further that m is continuous and does not vanish on $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Then, $m^{-1} \in M^p$ for every $p \in (1,\infty)$.*

See [5, 2] for related results. Applying Theorem B, we can deduce the following (see [12, Theorem 3.8]).

Theorem C. *Suppose that g_ψ is bounded on L^p for every $p \in (1,\infty)$. Let $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2\,dt/t$. If m is continuous and strictly positive on S^{n-1}, then we have*

$$\|f\|_p \leq c_p\|g_\psi(f)\|_p,$$

and hence $\|f\|_p \simeq \|g_\psi(f)\|_p$, $f \in L^p$, for all $p \in (1,\infty)$.

In this note we shall generalize Theorems B and C to weighted L^p spaces with A_p weights of Muckenhoupt (see Theorems 2.5, 2.9 and Corollaries 2.6, 2.11). Our proof of Theorem 2.5 has some features in common with the proof of Wiener-Lévy theorem in [30, vol. I, Chap. VI]. We also consider a discrete parameter version of g_ψ:

$$\Delta_\psi(f)(x) = \left(\sum_{k=-\infty}^{\infty} |f * \psi_{2^k}(x)|^2 \right)^{1/2}. \qquad (1.9)$$

We shall have Δ_ψ analogues of results for g_ψ (see Theorem 3.5 and Corollary 3.7). We formulate Theorems 2.9 and 3.5 in general forms so that they include unweighted cases as special cases, while Corollaries 2.11 and 3.7 may be more convenient for some applications.

In the unweighted case, we shall prove some results on H^p analogous to Corollaries 2.11 and 3.7 for p close to 1, $p \leq 1$, in Section 4 under a certain regularity condition for ψ (Theorems 4.7 and 4.8). We shall consider functions ψ including those which cannot be treated directly by the theory of

[28]. As a result, in particular, we shall be able to give a proof of the second inequality of (1.7) for $1/2 < \alpha < 3/2$ and $2/(2\alpha + 1) < p \le 1$ by methods of real analysis which does not depend on the Poisson kernel.

Here we recall some more background materials on μ_α. When $p < 1$ and $1/2 < \alpha < 1$, we know proofs for the first and the second inequality of (1.7) which use pointwise relations $\mu_\alpha(f) \ge c g_0(f)$ and $\mu_\alpha(f) \sim g_\lambda^*(f)$ with $\lambda = 1 + 2\alpha$, respectively, and apply appropriate properties of g_0 and g_λ^*. Also, we note that a proof of the inequality $\|\mu(f)\|_1 \le C\|f\|_{H^1}$ using a theory of vector valued singular integrals can be found in [10, Chap. V] (see also [17]). We have assumed that $\operatorname{supp}(\hat{f}) \subset [0, \infty)$ in stating $\mu_\alpha(f) \sim g_\lambda^*(f)$ and $g_0(f)$, $g_\lambda^*(f)$ are the Littlewood-Paley functions defined by

$$g_0(f)(x) = \left(\int_0^\infty |(\partial/\partial x) u(x,t)|^2 t \, dt \right)^{1/2},$$

$$g_\lambda^*(f)(x) = \left(\iint_{\mathbb{R} \times (0,\infty)} \left(\frac{t}{t + |x - y|} \right)^\lambda |\nabla u(y,t)|^2 \, dy \, dt \right)^{1/2}$$

with $u(y,t)$ denoting the Poisson integral of f: $u(y,t) = P_t * f(y)$ (see [27], [19] and references therein, and also [13], [15] for related results).

In [28], a proof of $\|f\|_{H^p} \le C\|g_Q(f)\|_p$ on $\mathbb{R}^n$ is given without the use of harmonicity (see [9] for the original proof using properties of harmonic functions). Also, when $n = 1$, a similar result is shown for g_0. It is to be noted that, combining this with the pointwise relation between g_0 and μ_α mentioned above, we can give a proof of the first inequality of (1.7) for the whole range of p, α in such a manner that a special property of the Poisson kernel is used only to prove the pointwise relation.

In Section 5, we shall apply Corollaries 2.11 and 3.7 to the theory of Sobolev spaces. In [1], the operator

$$U_\alpha(f)(x) = \left(\int_0^\infty \left| f(x) - \fint_{B(x,t)} f(y) \, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0, \qquad (1.10)$$

was studied, where $\fint_{B(x,t)} f(y) \, dy$ is defined as $|B(x,t)|^{-1} \int_{B(x,t)} f(y) \, dy$ with $|B(x,t)|$ denoting the Lebesgue measure of a ball $B(x,t)$ in $\mathbb{R}^n$ of radius t centered at x. The operator U_1 was used to characterize the Sobolev space $W^{1,p}(\mathbb{R}^n)$.

Theorem D. *Let* $1 < p < \infty$. *Then, the following two statements are equivalent*:

(1) f *belongs to* $W^{1,p}(\mathbb{R}^n)$,
(2) $f \in L^p(\mathbb{R}^n)$ *and* $U_1(f) \in L^p(\mathbb{R}^n)$.

Furthermore, from either of the two conditions $(1), (2)$ *it follows that*

$$\|U_1(f)\|_p \simeq \|\nabla f\|_p.$$

This may be used to define a Sobolev space analogous to $W^{1,p}(\mathbb{R}^n)$ in metric measure spaces. We shall also consider a discrete parameter version of U_α:

$$E_\alpha(f)(x) = \left(\sum_{k=-\infty}^{\infty} \left| f(x) - \fint_{B(x,2^k)} f(y)\,dy \right|^2 2^{-2k\alpha} \right)^{1/2}, \quad \alpha > 0, \quad (1.11)$$

and prove an analogue of Theorem D for E_α. Further, we shall consider operators generalizing U_α, E_α and show that they can be used to characterize the weighted Sobolev spaces, focusing on the case $0 < \alpha < n$.

2. Invertibility of homogeneous Fourier multipliers and Littlewood-Paley operators

We say that a weight function w belongs to the weight class A_p, $1 < p < \infty$, of Muckenhoupt on $\mathbb{R}^n$ if

$$[w]_{A_p} = \sup_B \left(|B|^{-1} \int_B w(x)\,dx \right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)}\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. Also, we say that $w \in A_1$ if $M(w) \leq Cw$ almost everywhere, with M denoting the Hardy-Littlewood maximal operator

$$M(f)(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)|\,dy,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ containing x; we denote by $[w]_{A_1}$ the infimum of all such C.

Let $m \in L^\infty(\mathbb{R}^n)$ and $w \in A_p$, $1 < p < \infty$. Let T_m be as in (1.8). We say that m is a Fourier multiplier for L^p_w and write $m \in M^p(w)$ if there exists a constant $C > 0$ such that

$$\|T_m(f)\|_{p,w} \leq C\|f\|_{p,w} \qquad (2.1)$$

for all $f \in L^2 \cap L^p_w$, where

$$\|f\|_{p,w} = \|f\|_{L^p_w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \right)^{1/p}.$$

We also write $L^p(w)$ for L^p_w. Define

$$\|m\|_{M^p(w)} = \inf C,$$

where the infimum is taken over all the constants C satisfying (2.1). Since $L^2 \cap L^p_w$ is dense in L^p_w, T_m uniquely extends to a bounded linear operator on L^p_w if $m \in M^p(w)$. In this note we shall confine our attention to the case of L^p_w boundedness of T_m with $w \in A_p$. We note that $M^p(w) = M^{p'}(\widetilde{w}^{-p'/p})$ by duality, where $1/p + 1/p' = 1$ and $\widetilde{w}(x) = w(-x)$.

If $w \in A_p$, $1 < p < \infty$, then $w^s \in A_r$ for some $s > 1$ and $r < p$ (see [10]). In applying interpolation arguments it is useful if sets of those (r, s) are specially notated.

Definition 2.1. Let $w \in A_p$, $1 < p < \infty$. If $0 < \sigma < p - 1$, $\tau > 0$ and if $w^s \in A_r$ for all $r \in (p - \sigma, p + \sigma)$ and all $s \in [1, 1 + \tau)$, we define a set $U(w, p)$ by
$$U(w, p) = U(w, p, \sigma, \tau) = (p - \sigma, p + \sigma) \times [1, 1 + \tau).$$
Let $\mathcal{F}(w, p)$ be the family of all such $U(w, p)$. We write $m \in M(U(w, p))$ if $m \in M^r(w^s)$ for all $(r, s) \in U(w, p)$.

We need a relation of $\|m\|_{M^p(w)}$ and $\|m\|_\infty$ in the following.

Proposition 2.2. *Let* $w \in A_p$, $1 < p < \infty$. *Suppose that* $m \in M(U(w, p))$ *for some* $U(w, p) \in \mathcal{F}(w, p)$. *Then*

$\|m\|_{M^p(w)} \le \|m\|_\infty^{1-\theta} \|m\|_{M^{p+\delta}(w^{1+\epsilon})}^\theta$, $\quad (p + \delta, 1 + \epsilon) \in U(w, p)$, $\quad$ *if* $p > 2$;

$\|m\|_{M^p(w)} \le \|m\|_\infty^{1-\theta} \|m\|_{M^p(w^{1+\epsilon})}^\theta$, $\quad (p, 1 + \epsilon) \in U(w, p)$, $\quad$ *if* $p = 2$;

$\|m\|_{M^p(w)} \le \|m\|_\infty^{1-\theta} \|m\|_{M^{p-\delta}(w^{1+\epsilon})}^\theta$, $\quad (p - \delta, 1 + \epsilon) \in U(w, p)$, $\quad$ *if* $1 < p < 2$,

for some $\theta \in (0, 1)$ *and some small numbers* $\delta, \epsilon > 0$.

Proof. Let $1 < p < 2$ and $w \in A_p$. Then, there exist $\epsilon_0, \delta_0 > 0$ such that $(p - \delta, 1 + \epsilon) \in U(w, p)$ for all $\epsilon \in (0, \epsilon_0]$ and $\delta \in (0, \delta_0]$. Let $1/p = (1 - \theta)/2 + \theta/(p - \delta)$, $\delta \in (0, \delta_0]$. Then, since $m \in M^{p-\delta}(w^{1+\epsilon})$, by interpolation with change of measures of Stein-Weiss (see [3]) between L^2 and $L^{p-\delta}(w^{1+\epsilon})$ boundedness, we have
$$\|m\|_{M^p(w^{p\theta(1+\epsilon)/(p-\delta)})} \le \|m\|_\infty^{1-\theta} \|m\|_{M^{p-\delta}(w^{1+\epsilon})}^\theta.$$
Note that $p\theta/(p - \delta) = (2 - p)/(2 - p + \delta)$. Thus we can choose $\epsilon, \delta > 0$ so that $p\theta(1 + \epsilon)/(p - \delta) = 1$. This completes the proof for $p \in (1, 2)$.

Suppose that $2 < p < \infty$ and $w \in A_p$. Then there exist $\epsilon_0 > 0$ $\delta_0 > 0$ such that $(p + \delta, 1 + \epsilon) \in U(w, p)$ for all $\epsilon \in (0, \epsilon_0]$ and $\delta \in (0, \delta_0]$. So, $m \in M^{p+\delta}(w^{1+\epsilon})$ for all $\epsilon \in (0, \epsilon_0]$ and $\delta \in (0, \delta_0]$. Similarly to the case $1 < p < 2$, applying interpolation, we have
$$\|m\|_{M^p(w^{p\theta(1+\epsilon)/(p+\delta)})} \le \|m\|_\infty^{1-\theta} \|m\|_{M^{p+\delta}(w^{1+\epsilon})}^\theta$$
with $p\theta/(p + \delta) = (p - 2)/(p + \delta - 2)$. Taking ϵ, δ so that $p\theta(1 + \epsilon)/(p + \delta) = 1$, we conclude the proof for $p \in (2, \infty)$. The case $p = 2$ can be handled similarly. $\qquad \square$

To treat Fourier multipliers arising from Littlewood-Paley functions in (1.2) and (1.9) simultaneously, we slightly generalize the usual notion of homogeneity.

Definition 2.3. Let f be a function on $\mathbb{R}^n$. We say that f is dyadically homogeneous of degree τ, $\tau \in \mathbb{R}$, if $f(2^k x) = 2^{k\tau} f(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $k \in \mathbb{Z}$ (the set of integers).

For $m \in M_w^p$, $1 < p < \infty$, $w \in A_p$, we consider the spectral radius operator

$$\rho_{p,w}(m) = \lim_{k \to \infty} \|m^k\|_{M^p(w)}^{1/k}.$$

To prove a weighted version of Theorem B, we need an approximation result for Fourier multipliers in $M^p(w)$.

Proposition 2.4. *Let $1 < p < \infty$, $w \in A_p$ and $m \in L^\infty(\mathbb{R}^n)$. We assume that m is dyadically homogeneous of degree 0 and continuous on the closed annulus $B_0 = \{\xi \in \mathbb{R}^n : 1 \le |\xi| \le 2\}$. We further assume that there exists $U(w,p) \in \mathcal{F}(w,p)$ such that $m \in M(U(w,p))$. Then, for any $\epsilon > 0$, there exists $\ell \in M^p(w)$ which is dyadically homogeneous of degree 0 and in $C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\|m - \ell\|_\infty < \epsilon$ and $\rho_{p,w}(m - \ell) < \epsilon$.*

Proof. Let $\{\varphi_j\}_{j=1}^\infty$ be a sequence of functions on the orthogonal group $O(n)$ such that

- each φ_j is infinitely differentiable and non-negative,
- for any neighborhood U of the identity in $O(n)$, there exists a positive integer N such that $\mathrm{supp}(\varphi_j) \subset U$ if $j \ge N$,
- $\int_{O(n)} \varphi_j(A)\, dA = 1$, where dA is the Haar measure on $O(n)$.

Also, let $\{\psi_j\}_{j=1}^\infty$ be a sequence of non-negative functions in $C^\infty(\mathbb{R})$ such that $\mathrm{supp}(\psi_j) \subset [1 - 2^{-j}, 1 + 2^{-j}]$ and $\int_0^\infty \psi_j(t)\, dt/t = 1$. Define

$$m_j(\xi) = \int_0^\infty \int_{O(n)} m(tA\xi)\varphi_j(A)\psi_j(t)\, dA\, \frac{dt}{t}.$$

Then m_j is dyadically homogeneous of degree 0, infinitely differentiable and $m_j \to m$ uniformly in $\mathbb{R}^n \setminus \{0\}$ by the continuity of m on B_0. This can be shown similarly to [12, pp. 123-124], where we can find the case when m is homogeneous of degree 0. Also, for a positive integer k, the derivatives of m_j^k satisfy

$$|\partial_\xi^\gamma m_j(\xi)^k| \le C_{j,k,M}\|m\|_\infty^k |\xi|^{-|\gamma|}, \quad \partial_\xi^\gamma = (\partial/\partial\xi_1)^{\gamma_1} \dots (\partial/\partial\xi_n)^{\gamma_n} \qquad (2.2)$$

for all multi-indices γ with $|\gamma| \le M$, where M is any positive integer, $\gamma = (\gamma_1, \dots, \gamma_n)$, $|\gamma| = \gamma_1 + \dots + \gamma_n$, $\gamma_j \in \mathbb{Z}$, $\gamma_j \ge 0$ and we have $C_{j,k,M} \le C_{j,M}k^M$ with a constant $C_{j,M}$ independent of k. By (2.2), if M is sufficiently large, it follows that

$$\|m_j^k\|_{M^p(w)} \le CC_{j,k,M}\|m\|_\infty^k, \quad w \in A_p, 1 < p < \infty$$

with a constant C independent of k, which is well-known (see [4, 14] for related results). Thus, by the evaluation of $C_{j,k,M}$ we have

$$\rho_{p,w}(m_j) \le \|m\|_\infty. \qquad (2.3)$$

Since $m, m_j \in M(U(w,p))$, by Proposition 2.2, we can find r close to p, $s > 1$ with $(r,s) \in U(w,p)$ and $\theta \in (0,1)$ such that

$$\|(m - m_j)^k\|_{M^p(w)} \le \|(m - m_j)^k\|_\infty^{1-\theta}\|(m - m_j)^k\|_{M^r(w^s)}^\theta.$$

It follows that

$$\rho_{p,w}(m - m_j) \leq \|m - m_j\|_\infty^{1-\theta} \rho_{r,w^s}(m - m_j)^\theta.$$

Thus, by (2.3) we have

$$\rho_{p,w}(m - m_j) \leq \|m - m_j\|_\infty^{1-\theta} (\rho_{r,w^s}(m)^\theta + \rho_{r,w^s}(m_j)^\theta)$$
$$\leq \|m - m_j\|_\infty^{1-\theta} (\rho_{r,w^s}(m)^\theta + \|m\|_\infty^\theta).$$

This completes the proof since $\|m - m_j\|_\infty \to 0$ as $j \to \infty$. $\square$

Applying Proposition 2.4, we can generalize Theorem B as follows.

Theorem 2.5. *Suppose that $1 < p < \infty, w \in A_p$ and that $m \in L^\infty(\mathbb{R}^n)$ fulfills the hypotheses of Proposition 2.4. Also, suppose that $m(\xi) \neq 0$ for every $\xi \neq 0$. Let $\varphi(z)$ be holomorphic in $\mathbb{C} \setminus \{0\}$. Then we have $\varphi(m(\xi)) \in M^p(w)$.*

Proof. Define $\epsilon_0 > 0$ by

$$4\epsilon_0 = \min_{\xi \in \mathbb{R}^n \setminus \{0\}} |m(\xi)| = \min_{1 \leq |\xi| \leq 2} |m(\xi)|.$$

By Proposition 2.4, there is $\ell \in M^p(w)$ which is dyadically homogeneous of degree 0 and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$ such that $\|m - \ell\|_\infty < \epsilon_0$ and $\rho_{p,w}(m - \ell) < \epsilon_0$. If we consider a curve $C : \ell(\xi) + 2\epsilon_0 e^{i\theta}, 0 \leq \theta \leq 2\pi$, Cauchy's formula can be applied to represent $\varphi(m(\xi))$ by a contour integral as follows:

$$\varphi(m(\xi)) = \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta)}{\zeta - m(\xi)} \, d\zeta = \frac{\epsilon_0}{\pi} \int_0^{2\pi} \frac{\varphi(\ell(\xi) + 2\epsilon_0 e^{i\theta})}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} e^{i\theta} \, d\theta, \quad \xi \neq 0.$$

Note that

$$\frac{e^{i\theta}}{2\epsilon_0 e^{i\theta} + \ell(\xi) - m(\xi)} = \frac{1}{2\epsilon_0} \sum_{k=0}^\infty \left(\frac{m(\xi) - \ell(\xi)}{2\epsilon_0 e^{i\theta}} \right)^k;$$

the series converges uniformly in $\theta \in [0, 2\pi]$ since $|m(\xi) - \ell(\xi)| < \epsilon_0$. Thus

$$\varphi(m(\xi)) = \frac{1}{2\pi} \sum_{k=0}^\infty \left(\frac{m(\xi) - \ell(\xi)}{2\epsilon_0} \right)^k M_k(\xi)$$

uniformly in $\mathbb{R}^n \setminus \{0\}$, where

$$M_k(\xi) = \int_0^{2\pi} \varphi(\ell(\xi) + 2\epsilon_0 e^{i\theta}) e^{-ik\theta} \, d\theta.$$

Since $|\ell(\xi) + 2\epsilon_0 e^{i\theta}| \geq \epsilon_0$, we can see that $M_k(\xi)$ is dyadically homogeneous of degree 0 and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$; also the derivative satisfies

$$|\partial_\xi^\gamma M_k(\xi)| \leq C_\gamma |\xi|^{-|\gamma|}$$

for every multi-index γ with a constant C_γ independent of k, which follows from homogeneity. This implies that $\|M_k\|_{M^p(w)} \leq C$ with a constant C

independent of k, similarly to the estimate for $\|m_j^k\|_{M^p(w)}$ in the proof of Proposition 2.4. Thus we have $\varphi(m) \in M^p(w)$ and

$$\|\varphi(m)\|_{M^p(w)} \le \frac{1}{2\pi} \sum_{k=0}^{\infty} (2\epsilon_0)^{-k} \|(m-\ell)^k\|_{M^p(w)} \|M_k\|_{M^p(w)},$$

since the series converges, for $\|(m-\ell)^k\|_{M^p(w)} \le \epsilon_0^k$ if k is sufficiently large. This completes the proof.

$\square$

Theorem 2.5 in particular implies the following.

Corollary 2.6. *Let $1 < p < \infty$ and $w \in A_p$. Let m be a dyadically homogeneous function of degree 0 such that $m \in M^r(v)$ for all $r \in (1, \infty)$ and all $v \in A_r$. We assume that m is continuous on B_0 and does not vanish there. Then $m^{-1} \in M^p(w)$.*

We have applications of Theorem 2.5 and Corollary 2.6 to the theory of Littlewood-Paley operators. Let $w \in A_p$, $1 < p < \infty$. We say that g_ψ of (1.2) is bounded on L_w^p if there exists a constant C such that $\|g_\psi(f)\|_{p,w} \le C\|f\|_{p,w}$ for $f \in L_w^2 \cap L^2$. The unique sublinear extension on L_w^p is also denoted by g_ψ. The L_w^p boundedness for Δ_ψ of (1.9) is considered similarly.

Let $\mathcal{H}$ be the Hilbert space of functions $u(t)$ on $(0, \infty)$ such that $\|u\|_{\mathcal{H}} = \left(\int_0^\infty |u(t)|^2 \, dt/t \right)^{1/2} < \infty$. We consider weighted spaces $L_{w,\mathcal{H}}^p$ of functions $h(y,t)$ with the norm

$$\|h\|_{p,w,\mathcal{H}} = \left(\int_{\mathbb{R}^n} \|h^y\|_{\mathcal{H}}^p w(y) \, dy \right)^{1/p},$$

where $h^y(t) = h(y,t)$. If $w = 1$ identically, the spaces $L_{w,\mathcal{H}}^p$ will be written simply as $L_{\mathcal{H}}^p$.

Define

$$E_\psi^\epsilon(h)(x) = \int_0^\infty \int_{\mathbb{R}^n} \psi_t(x-y) h_{(\epsilon)}(y,t) \, dy \, \frac{dt}{t}, \qquad (2.4)$$

where $h \in L_{\mathcal{H}}^2$ and $h_{(\epsilon)}(y,t) = h(y,t)\chi_{(\epsilon,\epsilon^{-1})}(t)$, $0 < \epsilon < 1$, and we assume that $\psi \in L^1(\mathbb{R}^n)$ with (1.1).

Then we have the following.

Lemma 2.7. *Let $1 < r < \infty$ and $v \in A_r$. We assume that*

$$\|g_\psi(f)\|_{r', v^{-r'/r}} \le C_0(r,v)\|f\|_{r', v^{-r'/r}}.$$

Then, if $h \in L_{v,\mathcal{H}}^r \cap L_{\mathcal{H}}^2$, we have

$$\sup_{\epsilon \in (0,1)} \|E_{\bar{\psi}}^\epsilon(h)\|_{r,v} \le C_0(r,v)\|h\|_{r,v,\mathcal{H}},$$

where $\bar{\psi}$ denotes the complex conjugate.

Proof. For $f \in \mathcal{S}(\mathbb{R}^n)$, we see that

$$\left| \int_{\mathbb{R}^n} E^\epsilon_{\widetilde{\psi}}(h)(x) f(x)\, dx \right| = \left| \int_{\mathbb{R}^n} \left(\int_\epsilon^{\epsilon^{-1}} \int_{\mathbb{R}^n} \widetilde{\psi}_t(x-y) h(y,t)\, dy\, \frac{dt}{t} \right) f(x)\, dx \right|$$

$$= \left| \int_\epsilon^{\epsilon^{-1}} \int_{\mathbb{R}^n} \bar{\psi}_t * f(y) h(y,t)\, dy\, \frac{dt}{t} \right|$$

$$\leq \int_{\mathbb{R}^n} g_\psi(\bar{f})(y) \|h^y\|_{\mathcal{H}}\, dy.$$

Thus, by Hölder's inequality, we have

$$\left| \int_{\mathbb{R}^n} E^\epsilon_{\widetilde{\psi}}(h)(x) f(x)\, dx \right| \leq \|g_\psi(\bar{f})\|_{r', v^{-r'/r}} \left(\int \|h^y\|^r_{\mathcal{H}} v(y)\, dy \right)^{1/r}$$

$$\leq C_0(r,v) \|f\|_{r', v^{-r'/r}} \left(\int \|h^y\|^r_{\mathcal{H}} v(y)\, dy \right)^{1/r}.$$

Taking the supremum over f with $\|f\|_{r', v^{-r'/r}} \leq 1$, we get the desired result.
$\square$

By applying Lemma 2.7, we have the following.

Proposition 2.8. *Suppose that* g_ψ *satisfies the hypothesis of Lemma 2.7 with* $r \in (1, \infty)$ *and* $v \in A_r$. *Also, we assume that*

$$\|g_\psi(f)\|_{r,v} \leq C_1(r,v) \|f\|_{r,v}.$$

Put

$$m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2\, \frac{dt}{t}.$$

Then $\|m\|_{M^r(v)} \leq C_0(r,v) C_1(r,v)$.

Proof. We first note that an interpolation with change of measures between the $L^r(v)$ and $L^{r'}(v^{-r'/r})$ boundedness of g_ψ implies the L^2 boundedness of g_ψ. Thus we have $m \in L^\infty(\mathbb{R}^n)$.

Let $F(y,t) = f * \psi_t(y)$, $f \in L^p_w \cap L^2$. Then

$$E^\epsilon_{\widetilde{\psi}}(F)(x) = \int_\epsilon^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t * f(y) \bar{\psi}_t(y-x)\, dy\, \frac{dt}{t} = \int_{\mathbb{R}^n} \Psi^{(\epsilon)}(x-z) f(z)\, dz,$$

where

$$\Psi^{(\epsilon)}(x) = \int_\epsilon^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t(x+y) \bar{\psi}_t(y)\, dy\, \frac{dt}{t}.$$

We see that

$$\widehat{\Psi^{(\epsilon)}}(\xi) = \int_\epsilon^{\epsilon^{-1}} \hat{\psi}(t\xi) \widehat{\bar{\psi}}(-t\xi)\, \frac{dt}{t} = \int_\epsilon^{\epsilon^{-1}} |\hat{\psi}(t\xi)|^2\, \frac{dt}{t}.$$

Thus

$$\int_{\mathbb{R}^n} \Psi^{(\epsilon)}(x-z)f(z)\,dz = T_{m^{(\epsilon)}}f(x), \quad m^{(\epsilon)}(\xi) = \int_{\epsilon}^{\epsilon^{-1}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t}.$$

From Lemma 2.7 and the L_v^r boundedness of g_ψ it follows that

$$\|T_{m^{(\epsilon)}}f\|_{r,v} = \|E_{\tilde{\psi}}^{\epsilon}(F)\|_{r,v} \le C_0(r,v)\|g_\psi(f)\|_{r,v} \le C_0(r,v)C_1(r,v)\|f\|_{r,v}.$$

(2.5)

Letting $\epsilon \to 0$ and noting $m^{(\epsilon)} \to m$, we see that $m \in M^r(v)$ and $\|m\|_{M^r(v)}$ can be evaluated by (2.5). $\qquad\square$

Now we can state a weighted version of Theorem C.

Theorem 2.9. *Let g_ψ be as in (1.2). Let $w \in A_p$, $1 < p < \infty$. Suppose that there exists $U(w,p) \in \mathfrak{F}(w,p)$ such that g_ψ fulfills the hypotheses of Proposition 2.8 on the weighted boundedness for all r, $v = w^s$, $(r,s) \in U(w,p)$. Further, suppose that $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2\,dt/t$ is continuous and does not vanish on S^{n-1}. Then we have*

$$\|f\|_{p,w} \le C_{p,w}\|g_\psi(f)\|_{p,w}$$

for $f \in L_w^p$.

Obviously, this implies Theorem C when $w = 1$.

Proof of Theorem 2.9. We first note that by Proposition 2.8 $m \in M(U(w,p))$. Thus from Theorem 2.5 with $\varphi(z) = 1/z$ and our assumptions, we see that $m^{-1} \in M^p(w)$. Since $f = T_{m^{-1}}T_m f$, $f \in L_w^p \cap L^2$, we have

$$\|f\|_{p,w} \le C\|T_m f\|_{p,w}.$$

Also, by (2.5) it follows that

$$\|T_m f\|_{p,w} \le C\|g_\psi(f)\|_{p,w}.$$

Combining results we have the desired inequality. $\qquad\square$

From Theorem 2.9 the next result follows.

Theorem 2.10. *Suppose the following.*
(1) $\|g_\psi(f)\|_{r,v} \le C_{r,v}\|f\|_{r,v}$ *for all $r \in (1,\infty)$ and all $v \in A_r$;*
(2) $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2\,dt/t$ *is continuous and strictly positive on S^{n-1}.*
Then, if $f \in L_w^p$, we have

$$\|f\|_{p,w} \le C_{p,w}\|g_\psi(f)\|_{p,w}$$

for all $p \in (1,\infty)$ and $w \in A_p$.

The following result is known (see [18]).

Theorem E. *Suppose that*
(1) $B_\epsilon(\psi) < \infty$ *for some $\epsilon > 0$, where $B_\epsilon(\psi) = \int_{|x|>1} |\psi(x)|\,|x|^\epsilon\,dx$;*
(2) $C_u(\psi) < \infty$ *for some $u > 1$ with $C_u(\psi) = \int_{|x|<1} |\psi(x)|^u\,dx$;*
(3) $H_\psi \in L^1(\mathbb{R}^n)$, *where $H_\psi(x) = \sup_{|y|\ge|x|} |\psi(y)|$.*

Then

$$\|g_\psi(f)\|_{p,w} \leq C_{p,w}\|f\|_{p,w}$$

for all $p \in (1, \infty)$ and $w \in A_p$.

By Theorem 2.10 and Theorem E we have the following result, which is useful in some applications.

Corollary 2.11. *Suppose that ψ satisfies the conditions $(1), (2), (3)$ of Theorem E and the non-degeneracy condition:* $\sup_{t>0} |\hat{\psi}(t\xi)| > 0$ *for all $\xi \neq 0$. Then $\|f\|_{p,w} \simeq \|g_\psi(f)\|_{p,w}$, $f \in L_w^p$, for all $p \in (1, \infty)$ and $w \in A_p$.*

Proof. Let $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 \, dt/t$. Then by our assumption $m(\xi) \neq 0$ for $\xi \neq 0$. Thus by Theorem E and Theorem 2.10, it suffices to show that m is continuous on S^{n-1}. From [18] we have

$$\int_{2^k}^{2^{k+1}} |\hat{\psi}(t\xi)|^2 \, \frac{dt}{t} \leq C \min\left(2^{k\epsilon}, 2^{-k\epsilon}\right)$$

with some $\epsilon > 0$ for $\xi \in S^{n-1}$ and $k \in \mathbb{Z}$. Thus it can be seen that $\int_\epsilon^{\epsilon^{-1}} |\hat{\psi}(t\xi)|^2 \, dt/t \to m(\xi)$ uniformly on S^{n-1} as $\epsilon \to 0$. Since $\int_\epsilon^{\epsilon^{-1}} |\hat{\psi}(t\xi)|^2 \, dt/t$ is continuous on S^{n-1} for each fixed $\epsilon > 0$, the continuity of m follows. $\square$

Remark 2.12. Let $1 < p < \infty$, $w \in A_p$. Suppose that g_ψ is bounded on L_w^p and ψ is a radial function with $\int_0^\infty |\hat{\psi}(t\xi)|^2 \, dt/t = 1$ for every $\xi \neq 0$. Then we have $\|f\|_{p,w} \leq C\|g_\psi(f)\|_{p,w}$ if g_ψ is also bounded on $L_{w^{-p'/p}}^{p'}$. This is well-known and follows from the proofs of Lemma 2.7 and Proposition 2.8. Also, this can be proved by applying arguments of [10, Chap. V, 5.6 (b)].

3. Discrete parameter Littlewood-Paley functions

Let $\psi \in L^1(\mathbb{R}^n)$ with (1.1) and let Δ_ψ be as in (1.9). We first give a criterion for the boundedness of Δ_ψ on L_w^p analogous to Theorem E.

Theorem 3.1. *Let $B_\epsilon(\psi)$, H_ψ be as in Theorem E. Suppose that*

(1) $B_\epsilon(\psi) < \infty$ *for some $\epsilon > 0$;*
(2) $|\hat{\psi}(\xi)| \leq C|\xi|^{-\delta}$ *for all $\xi \in \mathbb{R}^n \setminus \{0\}$ with some $\delta > 0$;*
(3) $H_\psi \in L^1(\mathbb{R}^n)$.

Let $1 < p < \infty$. Then

$$\|\Delta_\psi(f)\|_{p,w} \leq C_{p,w}\|f\|_{p,w}$$

for every $w \in A_p$.

We assume the pointwise estimate of $\hat{\psi}$ in (2), which is not required in Theorem E.

Proof of Theorem 3.1. We apply methods of [7]. Define

$$\widehat{D_j(f)}(\xi) = \Psi(2^j\xi)\hat{f}(\xi) \qquad \text{for} \quad j \in \mathbb{Z},$$

where $\Psi \in C^\infty$ satisfies that $\text{supp}(\Psi) \subset \{1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{j=-\infty}^{\infty} \Psi(2^j\xi) = 1 \qquad \text{for} \quad \xi \neq 0.$$

We write

$$f * \psi_{2^k}(x) = \sum_{j=-\infty}^{\infty} D_{j+k}(f * \psi_{2^k})(x),$$

where we initially assume that $f \in \mathcal{S}(\mathbb{R}^n)$. Let

$$L_j(f)(x) = \left(\sum_{k=-\infty}^{\infty} |D_{j+k}(f * \psi_{2^k})(x)|^2 \right)^{1/2}.$$

Then

$$\Delta_\psi(f)(x) \leq \sum_{j \in \mathbb{Z}} L_j(f)(x).$$

We note that the condition (1) and (1.1) imply that $|\hat{\psi}(\xi)| \leq C|\xi|^{\epsilon'}$, $\epsilon' = \min(1, \epsilon)$ (see [18, Lemma 1]). So, since the Fourier transform of $D_j(f * \psi_{2^k})$ is supported in $E_j = \{2^{-1-j} \leq |\xi| \leq 2^{1-j}\}$, the Plancherel theorem and the conditions (1), (2) imply that

$$\begin{aligned}
\|L_j(f)\|_2^2 &= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} |D_{j+k}(f * \psi_{2^k})(x)|^2 \, dx \qquad\qquad (3.1)\\
&\leq \sum_{k \in \mathbb{Z}} C \int_{E_{j+k}} \min\left(|2^k\xi|^\epsilon, |2^k\xi|^{-\epsilon}\right) \left|\hat{f}(\xi)\right|^2 \, d\xi \\
&\leq C 2^{-\epsilon|j|} \sum_{k \in \mathbb{Z}} \int_{E_{j+k}} \left|\hat{f}(\xi)\right|^2 \, d\xi \\
&\leq C 2^{-\epsilon|j|} \|f\|_2^2
\end{aligned}$$

for some $\epsilon > 0$, where to get the last inequality we also use the fact that the sets E_j are finitely overlapping.

By the condition (3), we see that $\sup_{t>0} |f * \psi_t| \leq CM(f)$ (see [24, pp. 63–64]). Thus, if $w \in A_2$, by the Hardy-Littlewood maximal theorem and the

Littlewood-Paley inequality for L_w^2 we see that

$$\|L_j(f)\|_{2,w}^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} |D_{j+k}(f) * \psi_{2^k}(x)|^2 \ w(x)\, dx \qquad (3.2)$$

$$\leq \sum_{k \in \mathbb{Z}} C \int_{\mathbb{R}^n} |M(D_{j+k}(f))(x)|^2 \ w(x)\, dx$$

$$\leq \sum_{k \in \mathbb{Z}} C \int_{\mathbb{R}^n} |D_{j+k}(f)(x)|^2 \ w(x)\, dx$$

$$\leq C \|f\|_{2,w}^2.$$

Interpolation with change of measures between (3.1) and (3.2) implies that

$$\|L_j(f)\|_{2,w^u} \leq C2^{-\epsilon(1-u)|j|/2} \|f\|_{2,w^u}$$

for $u \in (0,1)$. Choosing u, close to 1, so that $w^{1/u} \in A_2$, we have

$$\|L_j(f)\|_{2,w} \leq C2^{-\epsilon(1-u)|j|/2} \|f\|_{2,w},$$

and hence

$$\|\Delta_\psi(f)\|_{2,w} \leq \sum_{j \in \mathbb{Z}} \|L_j(f)\|_{2,w} \leq C \|f\|_{2,w}.$$

Thus the conclusion follows from the extrapolation theorem of Rubio de Francia [16]. □

Remark 3.2. Under the hypotheses of Theorem 3.1, g_ψ is also bounded on L_w^p for all $p \in (1, \infty)$ and $w \in A_p$. This can be seen from the proof of Theorem E in [18].

Let $\mathcal{K}$ be the Hilbert space of functions $v(k)$ on $\mathbb{Z}$ such that

$$\|v\|_{\mathcal{K}} = \left(\sum_{k=-\infty}^{\infty} |v(k)|^2 \right)^{1/2} < \infty.$$

We define spaces $L_{w,\mathcal{K}}^p$ similarly to $L_{w,\mathcal{H}}^p$. Also, we use notation similar to the one used when $E_\psi^\epsilon(h)$ is considered. We define

$$L_\psi^N(l)(x) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \psi_{2^k}(x-y) l_{(N)}(y,k)\, dy, \qquad (3.3)$$

where $l \in L_{\mathcal{K}}^2$, $l_{(N)}(x,k) = l(x,k)\chi_{[-N,N]}(k)$ for a positive integer N.

Then, we have the following result.

Lemma 3.3. *Suppose that* $1 < r < \infty$, $v \in A_r$ *and that*

$$\|\Delta_\psi(f)\|_{r',v^{-r'/r}} \leq C_0(r,v) \|f\|_{r',v^{-r'/r}}.$$

Then, we have $\sup_{N \geq 1} \|L_{\bar\psi}^N(l)\|_{r,v} \leq C_0(r,v) \|l\|_{r,v,\mathcal{K}}$, *that is,*

$$\sup_{N \geq 1} \left(\int_{\mathbb{R}^n} |L_{\bar\psi}^N(l)(x)|^r v(x)\, dx \right)^{1/r} \leq C_0(r,v) \left(\int_{\mathbb{R}^n} \left(\sum_{k=-\infty}^{\infty} |l(x,k)|^2 \right)^{r/2} v(x)\, dx \right)^{1/r}$$

for $l \in L^r_{v,\mathcal{K}} \cap L^2_{\mathcal{K}}$.

This is used to prove the following.

Proposition 3.4. *We assume that Δ_ψ satisfies the hypothesis of Lemma 3.3 with $r \in (1,\infty)$ and $v \in A_r$. Further, we assume that*

$$\|\Delta_\psi(f)\|_{r,v} \leq C_1(r,v)\|f\|_{r,v}.$$

Set

$$m(\xi) = \sum_{k=-\infty}^{\infty} |\hat{\psi}(2^k \xi)|^2.$$

Then, we have $\|m\|_{M^r(v)} \leq C_0(r,v)C_1(r,v)$.

Proposition 3.4 and Theorem 2.5 are applied to prove the following.

Theorem 3.5. *We assume that $w \in A_p$, $1 < p < \infty$. Suppose that there exists $U(w,p) \in \mathcal{F}(w,p)$ such that Δ_ψ fulfills the hypotheses of Proposition 3.4 on the weighted boundedness for all r, $v = w^s$, $(r,s) \in U(w,p)$. Then if the function $m(\xi) = \sum_{k=-\infty}^{\infty} |\hat{\psi}(2^k\xi)|^2$ is continuous and does not vanish on B_0, we have*

$$\|f\|_{p,w} \leq C_{p,w}\|\Delta_\psi(f)\|_{p,w}$$

for $f \in L^p_w$.

We note that m is dyadically homogeneous of degree 0 and that, under the assumptions of Theorem 3.5, $m \in M(U(w,p))$.

Theorem 3.5 implies the next result.

Theorem 3.6. *We assume the following.*

(1) $\|\Delta_\psi(f)\|_{r,v} \leq C_{r,v}\|f\|_{r,v}$ *for all $r \in (1,\infty)$ and all $v \in A_r$;*

(2) m *is continuous and strictly positive on B_0, where m is defined as in Theorem 3.5.*

Let $w \in A_p$, $1 < p < \infty$. Then we have

$$\|f\|_{p,w} \leq C_{p,w}\|\Delta_\psi(f)\|_{p,w}, \quad f \in L^p_w.$$

Lemma 3.3, Proposition 3.4, Theorem 3.5 and Theorem 3.6 are analogous to and can be proved similarly to Lemma 2.7, Proposition 2.8, Theorem 2.9 and Theorem 2.10, respectively. We omit their proofs.

We also have an analogue of Corollary 2.11.

Corollary 3.7. *Suppose that ψ satisfies the conditions $(1),(2),(3)$ of Theorem 3.1 and the non-degeneracy condition: $\sup_{k \in \mathbb{Z}} |\hat{\psi}(2^k\xi)| > 0$ for all $\xi \neq 0$. Then $\|f\|_{p,w} \simeq \|\Delta_\psi(f)\|_{p,w}$, $f \in L^p_w$, for all $p \in (1,\infty)$ and $w \in A_p$.*

Proof. By the assumption $m(\xi) = \sum_{k=-\infty}^{\infty} |\hat{\psi}(2^k\xi)|^2 > 0$ for $\xi \neq 0$. Therefore, by Theorem 3.6, to prove a reverse inequality of the conclusion of Theorem 3.1 it suffices to show that m is continuous on B_0. From the estimate $|\hat{\psi}(\xi)| \leq C \min(|\xi|^\epsilon, |\xi|^{-\epsilon})$ for some $\epsilon > 0$, which follows from (1) and (2) of Theorem 3.1 (see [18, Lemma 1]), it can be seen that $\sum_{k=-N}^{N} |\hat{\psi}(2^k\xi)|^2 \to m(\xi)$ uniformly on B_0 as $N \to \infty$. Since $\sum_{k=-N}^{N} |\hat{\psi}(2^k\xi)|^2$ is continuous on

B_0 for each fixed N, we can conclude that m is also continuous on B_0. This completes the proof. □

4. Littlewood-Paley operators on H^p, $0 < p \leq 1$, with p close to 1

Let $0 < p \leq 1$. We consider the Hardy space of functions on $\mathbb{R}^n$ with values in $\mathcal{H}$, which is denoted by $H^p_{\mathcal{H}}(\mathbb{R}^n)$. Choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \varphi(x)\,dx = 1$. Let $h \in L^2_{\mathcal{H}}(\mathbb{R}^n)$. We say $h \in H^p_{\mathcal{H}}(\mathbb{R}^n)$ if $\|h\|_{H^p_{\mathcal{H}}} = \|h^*\|_{L^p} < \infty$ with

$$h^*(x) = \sup_{s > 0} \left(\int_0^\infty |\varphi_s * h^t(x)|^2 \, \frac{dt}{t} \right)^{1/2},$$

where we write $h^t(x) = h(x, t)$. Similarly, we consider the Hardy space $H^p_{\mathcal{K}}(\mathbb{R}^n)$ of functions l in $L^2_{\mathcal{K}}(\mathbb{R}^n)$ such that $\|l\|^p_{H^p_{\mathcal{K}}} = \|l^*\|_{L^p} < \infty$, where

$$l^*(x) = \sup_{s > 0} \left(\sum_{j=-\infty}^\infty |\varphi_s * l^j(x)|^2 \right)^{1/2}, \quad l^j(x) = l(x, j).$$

Let $\psi \in L^1(\mathbb{R}^n)$ with (1.1) and let $E^\epsilon_\psi(h)$ be defined as in (2.4).

Theorem 4.1. *Suppose that*

(1) $\int_0^\infty |\hat{\psi}(t\xi)|^2 \, dt/t \leq C$ *with a constant C;*
(2) *there exists $\tau \in (0, 1]$ such that if $|x| > 2|y|$,*

$$\left(\int_0^\infty |\psi_t(x - y) - \psi_t(x)|^2 \, \frac{dt}{t} \right)^{1/2} \leq C \frac{|y|^\tau}{|x|^{n+\tau}}.$$

Then

$$\sup_{\epsilon \in (0,1)} \|E^\epsilon_\psi(h)\|_{H^p} \leq C\|h\|_{H^p_{\mathcal{H}}}$$

if $n/(n + \tau) < p \leq 1$, where $H^p = H^p(\mathbb{R}^n)$ is the ordinary Hardy space on $\mathbb{R}^n$.

Recall that we say $f \in \mathcal{S}'(\mathbb{R}^n)$ (the space of tempered distributions) belongs to $H^p(\mathbb{R}^n)$ if $\|f\|_{H^p} = \|f^*\|_p < \infty$, where $f^*(x) = \sup_{t>0} |\varphi_t * f(x)|$, with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int \varphi(x)\,dx = 1$ (see [9]).

We also have a similar result for $L^N_\psi(l)$.

Theorem 4.2. *Let $L^N_\psi(l)$ be defined as in (3.3). We assume the following conditions:*

(1) $\sum_{k=-\infty}^\infty |\hat{\psi}(2^k\xi)|^2 \leq C$ *with a constant C;*
(2) *if $|x| > 2|y|$, we have*

$$\left(\sum_{k=-\infty}^\infty |\psi_{2^k}(x - y) - \psi_{2^k}(x)|^2 \right)^{1/2} \leq C \frac{|y|^\tau}{|x|^{n+\tau}}$$

with some $\tau \in (0, 1]$.

Then

$$\sup_{N \geq 1} \|L_\psi^N(l)\|_{H^p} \leq C\|l\|_{H^p_{\mathcal{K}}} \quad \text{for } n/(n+\tau) < p \leq 1.$$

To prove these theorems we apply atomic decompositions.

Let a be a (p, ∞) atom in $H^p_{\mathcal{K}}(\mathbb{R}^n)$. Thus

(i) $\left(\int_0^\infty |a(x, t)|^2\, dt/t\right)^{1/2} \leq |Q|^{-1/p}$, where Q is a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes;

(ii) $\operatorname{supp}(a(\cdot, t)) \subset Q$ uniformly in $t > 0$, where Q is the same as in (i);

(iii) $\int_{\mathbb{R}^n} a(x, t) x^\gamma\, dx = 0$ for all $t > 0$ and $|\gamma| \leq [n(1/p - 1)]$, where $x^\gamma = x_1^{\gamma_1} \ldots x_n^{\gamma_n}$ and $[a]$ denotes the largest integer not exceeding a.

To prove Theorem 4.1 we use the following.

Lemma 4.3. *Let* $h \in L^2_{\mathcal{K}}(\mathbb{R}^n)$. *Suppose that* $h \in H^p_{\mathcal{K}}(\mathbb{R}^n)$. *Then there exist a sequence* $\{a_k\}$ *of* (p, ∞) *atoms in* $H^p_{\mathcal{K}}(\mathbb{R}^n)$ *and a sequence* $\{\lambda_k\}$ *of positive numbers such that* $h = \sum_{k=1}^\infty \lambda_k a_k$ *in* $H^p_{\mathcal{K}}(\mathbb{R}^n)$ *and in* $L^2_{\mathcal{K}}(\mathbb{R}^n)$, *and* $\sum_{k=1}^\infty \lambda_k^p \leq C\|h\|^p_{H^p_{\mathcal{K}}}$, *where* C *is a constant independent of* h.

See [10, 26] for the case of $H^p(\mathbb{R}^n)$; the vector valued case can be proved similarly. We apply Lemma 4.3 for $p \in (n/(n+1), 1]$. We also need the following.

Lemma 4.4. *Let* φ *be a non-negative* C^∞ *function on* $\mathbb{R}^n$ *supported in* $\{|x| < 1\}$ *which satisfies* $\int \varphi(x)\, dx = 1$. *Suppose that* $\psi \in L^1(\mathbb{R}^n)$ *satisfies the conditions* $(1), (2)$ *of Theorem 4.1. Let* $\Psi_{s,t} = \varphi_s * \psi_t$, $s, t > 0$. *Then, if* $|x| > 3|y|$, *we have*

$$\left(\int_0^\infty |\Psi_{s,t}(x-y) - \Psi_{s,t}(x)|^2\, \frac{dt}{t}\right)^{1/2} \leq C \frac{|y|^\tau}{|x|^{n+\tau}}$$

with a constant C *independent of* $s > 0$.

Proof. We note that

$$\Psi_{s,t}(x-y) - \Psi_{s,t}(x) = \int_{|z|<s} (\psi_t(x-y-z) - \psi_t(x-z))\varphi_s(z)\, dz.$$

Let $0 < s < |x|/4$. Then, if $|x| > 3|y|$ and $|z| < s$, we have $|x - z| \geq (3/4)|x| \geq 2|y|$. Thus by the Minkowski inequality and (2) of Theorem 4.1 we see that

$$\left(\int_0^\infty |\Psi_{s,t}(x-y) - \Psi_{s,t}(x)|^2\, \frac{dt}{t}\right)^{1/2} \tag{4.1}$$

$$\leq \int_{|z|<s} \left(\int_0^\infty |\psi_t(x-y-z) - \psi_t(x-z)|^2\, \frac{dt}{t}\right)^{1/2} \varphi_s(z)\, dz$$

$$\leq C \int_{|z|<s} \frac{|y|^\tau}{|x-z|^{n+\tau}} \varphi_s(z)\, dz$$

$$\leq C\|\varphi\|_1 \frac{|y|^\tau}{|x|^{n+\tau}}.$$

To deal with the case $s \geq |x|/4$, we write

$$\Psi_{s,t}(x-y) - \Psi_{s,t}(x) = \int \hat{\varphi}(s\xi)\hat{\psi}(t\xi)e^{2\pi i\langle x,\xi\rangle} \left(e^{-2\pi i\langle y,\xi\rangle} - 1\right) d\xi.$$

Applying Minkowski's inequality again and using (1) of Theorem 4.1, we see that

$$\left(\int_0^\infty |\Psi_{s,t}(x-y) - \Psi_{s,t}(x)|^2 \frac{dt}{t}\right)^{1/2} \tag{4.2}$$

$$\leq \int |\hat{\varphi}(s\xi)|2\pi|y||\xi| \left(\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t}\right)^{1/2} d\xi$$

$$\leq C|y| \int |\hat{\varphi}(s\xi)||\xi| \, d\xi$$

$$\leq C|y|s^{-n-1} \int |\hat{\varphi}(\xi)||\xi| \, d\xi$$

$$\leq C\frac{|y|^\tau}{|x|^{n+\tau}}.$$

By (4.1) and (4.2) we get the desired estimates.   □

Proof of Theorem 4.1. Let a be a (p,∞) atom in $H^p_{\mathcal{H}}(\mathbb{R}^n)$ supported on the cube Q of the definition of the atom. Let y_0 be the center of Q. Let $\widetilde{Q}$ be a concentric enlargement of Q such that $3|y - y_0| < |x - y_0|$ if $y \in Q$ and $x \in \mathbb{R}^n \setminus \widetilde{Q}$. Let φ be as in Lemma 4.4. Then, using Lemma 4.4, the properties of an atom and the Schwarz inequality, for $x \in \mathbb{R}^n \setminus \widetilde{Q}$ we have

$$\left|\varphi_s * E_\psi^\epsilon(a)(x)\right| = \left|\iint \left(\Psi_{s,t}(x-y) - \Psi_{s,t}(x-y_0)\right) a_{(\epsilon)}(y,t) \, dy \, \frac{dt}{t}\right|$$

$$\leq \int_Q \left(\int_0^\infty |\Psi_{s,t}(x-y) - \Psi_{s,t}(x-y_0)|^2 \frac{dt}{t}\right)^{1/2} \left(\int_0^\infty |a(y,t)|^2 \frac{dt}{t}\right)^{1/2} dy$$

$$\leq C|Q|^{-1/p} \int_Q \left(\int_0^\infty |\Psi_{s,t}(x-y) - \Psi_{s,t}(x-y_0)|^2 \frac{dt}{t}\right)^{1/2} dy$$

$$\leq C|Q|^{-1/p} \int_Q |y - y_0|^\tau |x - y_0|^{-n-\tau} \, dy$$

$$\leq C|Q|^{-1/p+1+\tau/n}|x - y_0|^{-n-\tau}.$$

Since $p > n/(n+\tau)$, we thus have

$$\int_{\mathbb{R}^n \setminus \widetilde{Q}} \sup_{s>0} \left|\varphi_s * E_\psi^\epsilon(a)(x)\right|^p dx$$

$$\leq C|Q|^{-1+p+p\tau/n} \int_{\mathbb{R}^n \setminus \widetilde{Q}} |x - y_0|^{-p(n+\tau)} \leq C. \tag{4.3}$$

The condition (1) implies the L^2 boundedness of g_ψ and hence by Lemma 2.7 we can see that

$$\sup_{\epsilon \in (0,1)} \|E_\psi^\epsilon(h)\|_2 \leq C\|h\|_{L_{\mathcal{H}}^2}, \quad h \in L_{\mathcal{H}}^2(\mathbb{R}^n).$$

So, by Hölder's inequality and the properties (i), (ii) of a, we get

$$\int_{\widetilde{Q}} \sup_{s>0} \left|\varphi_s * E_\psi^\epsilon(a)(x)\right|^p \, dx \leq C|Q|^{1-p/2} \left(\int_{\widetilde{Q}} |M(E_\psi^\epsilon(a))(x)|^2 \, dx\right)^{p/2} \quad (4.4)$$

$$\leq C|Q|^{1-p/2} \left(\int_Q \int_0^\infty |a(y,t)|^2 \frac{dt}{t} \, dy\right)^{p/2}$$

$$\leq C.$$

Combining (4.3) and (4.4), we have

$$\int_{\mathbb{R}^n} \sup_{s>0} \left|\varphi_s * E_\psi^\epsilon(a)(x)\right|^p \, dx \leq C. \quad (4.5)$$

Let $h \in H_{\mathcal{H}}^p(\mathbb{R}^n)$ and the decomposition $h = \sum_{k=1}^\infty \lambda_k a_k$ be as in Lemma 4.3. We easily see that $|\varphi_s * E_\psi^\epsilon(h)(x)| \leq C_\epsilon \|\psi\|_1 \|\varphi_s\|_2 \|h\|_{L_{\mathcal{H}}^2}$. Thus, since $h = \sum_{k=1}^\infty \lambda_k a_k$ is in $L_{\mathcal{H}}^2(\mathbb{R}^n)$, we have

$$\varphi_s * E_\psi^\epsilon(h)(x) = \sum_{k=1}^\infty \lambda_k \varphi_s * E_\psi^\epsilon(a_k)(x).$$

By this and (4.5) we can prove

$$\int_{\mathbb{R}^n} \sup_{s>0} \left|\varphi_s * E_\psi^\epsilon(h)(x)\right|^p \, dx \leq C\|h\|_{H_{\mathcal{H}}^p}^p.$$

This completes the proof.

$\square$

Theorem 4.2 can be shown similarly.

Also, we can prove the following mapping properties of g_ψ and Δ_ψ on $H^p(\mathbb{R}^n)$ in the same way.

Theorem 4.5. *Suppose that ψ fulfills the hypotheses of Theorem 4.1. We define $F(\psi, f)(x, t) = f * \psi_t(x)$. Then if $n/(n+\tau) < p \leq 1$,*

$$\|F(\psi, f)\|_{H_{\mathcal{H}}^p} \leq C\|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Theorem 4.6. *We assume that ψ fulfills the hypotheses of Theorem 4.2. Let $G(\psi, f)(x, k) = f * \psi_{2^k}(x)$. Then*

$$\|G(\psi, f)\|_{H_{\mathcal{H}}^p} \leq C\|f\|_{H^p}, \quad f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),$$

if $n/(n+\tau) < p \leq 1$.

Proof of Theorem 4.5. The proof is similar to that of Theorem 4.1. By the atomic decomposition, it suffices to show that $\|F(\psi, a)\|_{H^p_{\mathcal{H}}} \leq C$, where a is a (p, ∞) atom in $H^p(\mathbb{R}^n)$ such that $\|a\|_\infty \leq |Q|^{-1/p}$, supp$(a) \subset Q$ with a cube Q and $\int a = 0$. Let y_0 be the center of Q and let $\widetilde{Q}$, φ_s, $\Psi_{s,t}$ be as in the proof of Theorem 4.1. Then, using Minkowski's inequality and Lemma 4.4, for $x \in \mathbb{R}^n \setminus \widetilde{Q}$ we have

$$\left(\int_0^\infty |\varphi_s * \psi_t * a(x)|^2 \, \frac{dt}{t} \right)^{1/2}$$

$$= \left(\int_0^\infty \left| \int \left(\Psi_{s,t}(x - y) - \Psi_{s,t}(x - y_0) \right) a(y) \, dy \right|^2 \frac{dt}{t} \right)^{1/2}$$

$$\leq C|Q|^{-1/p} \int_Q \left(\int_0^\infty |\Psi_{s,t}(x - y) - \Psi_{s,t}(x - y_0)|^2 \, \frac{dt}{t} \right)^{1/2} dy$$

$$\leq C|Q|^{-1/p+1+\tau/n} |x - y_0|^{-n-\tau}.$$

Therefore, as in (4.3), for $p > n/(n + \tau)$, we have

$$\int_{\mathbb{R}^n \setminus \widetilde{Q}} \sup_{s>0} \left(\int_0^\infty |\varphi_s * \psi_t * a(x)|^2 \, \frac{dt}{t} \right)^{p/2} \leq C.$$

Since by the Minkowski inequality we easily see that

$$\sup_{s>0} \left(\int_0^\infty |\varphi_s * \psi_t * a(x)|^2 \, \frac{dt}{t} \right)^{1/2} \leq \sup_{s>0} \varphi_s * g_\psi(a)(x) \leq C M \left(g_\psi(a) \right)(x),$$

as in (4.4) we have

$$\int_{\widetilde{Q}} \sup_{s>0} \left(\int_0^\infty |\varphi_s * \psi_t * a(x)|^2 \, \frac{dt}{t} \right)^{p/2} \leq C.$$

Collecting results, we have the desired estimates.

$\square$

The proof of Theorem 4.6 is similar. Using Theorems 4.1, 4.5 and Theorems 4.2, 4.6, we can show analogues of Corollaries 2.11 and 3.7 for $p \leq 1$.

Theorem 4.7. *Suppose that ψ fulfills the hypotheses of Theorem 4.1. Put $m(\xi) = \int_0^\infty |\hat{\psi}(t\xi)|^2 \, dt/t$. We assume that m does not vanish in $\mathbb{R}^n \setminus \{0\}$ and $m \in C^k(\mathbb{R}^n \setminus \{0\})$, where k is a positive integer satisfying $k/n > 1/p - 1/2$, with $n/(n + \tau) < p \leq 1$. Then we have*

$$\|F(\psi, f)\|_{H^p_{\mathcal{H}}} \simeq \|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, where $F(\psi, f)$ is as in Theorem 4.5.

Theorem 4.8. *We assume that ψ fulfills the hypotheses of Theorem 4.2. Set $m(\xi) = \sum_{j=-\infty}^\infty |\hat{\psi}(2^j \xi)|^2$. Let $n/(n + \tau) < p \leq 1$. We assume that $m(\xi) \neq 0$*

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and $m \in C^k(\mathbb{R}^n \setminus \{0\})$ with a positive integer k as in Theorem 4.7. Let $G(\psi, f)$ be as in Theorem 4.6. Then we have

$$\|G(\psi, f)\|_{H^p_{\mathcal{K}}} \simeq \|f\|_{H^p}, \quad f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Proof of Theorem 4.7. By Theorem 4.5 we have $\|F(\psi, f)\|_{H^p_{\mathcal{K}}} \leq C\|f\|_{H^p}$. To prove the reverse inequality we note that $f = T_{m^{-1}} T_m f$. Since $m^{-1} \in C^k(\mathbb{R}^n \setminus \{0\})$ and it is homogeneous of degree 0, m^{-1} is a Fourier multiplier for H^p by [10, pp. 347–348]. Thus

$$\|f\|_{H^p} \leq C\|T_m f\|_{H^p} \leq C \liminf_{\epsilon \to 0} \|T_{m^{(\epsilon)}} f\|_{H^p}, \tag{4.6}$$

and by the proof of Proposition 2.8 we see that

$$T_{m^{(\epsilon)}} f = E_{\widetilde{\psi}}^\epsilon(F),$$

where $m^{(\epsilon)}$, F are defined as in the proof of Proposition 2.8. Thus by Theorem 4.1 we have

$$\|T_{m^{(\epsilon)}} f\|_{H^p} = \|E_{\widetilde{\psi}}^\epsilon(F)\|_{H^p} \leq C\|F(\psi, f)\|_{H^p_{\mathcal{K}}},$$

which combined with (4.6) implies the reverse inequality. $\square$

Theorem 4.8 can be proved similarly.

We note that Theorems 4.5 and 4.6 imply that $\|g_\psi(f)\|_p \leq C\|f\|_{H^p}$, $\|\Delta_\psi(f)\|_p \leq C\|f\|_{H^p}$. Under the assumptions of Theorems 4.7 and 4.8, the reverse inequalities, which would improve results, are not available at present stage of the research. For related results which can handle Littlewood-Paley operators like g_Q, we refer to [28].

Let $\varphi^{(\alpha)}$ on $\mathbb{R}^1$ be as in (1.6). Then we can show that

$$\left(\int_0^\infty |\varphi_t^{(\alpha)}(x - y) - \varphi_t^{(\alpha)}(x)|^2 \frac{dt}{t} \right)^{1/2} \leq C \frac{|y|^\sigma}{|x|^{1+\sigma}}, \quad \sigma = (2\alpha - 1)/2, \tag{4.7}$$

if $2|y| < |x|$, where $1/2 < \alpha < 3/2$. Also, it is not difficult to see that the condition (1) of Theorem 4.1 is valid for $\varphi^{(\alpha)}$. Thus, from Theorem 4.5 we in particular have the second inequality of (1.7) for $1/2 < \alpha < 3/2$, $2/(2\alpha + 1) < p \leq 1$. We shall give a proof of the estimate (4.7) in Section 6 for completeness.

5. Applications to the theory of Sobolev spaces

Let $0 < \alpha < n$ and

$$T_\alpha(f)(x) = \left(\int_0^\infty \left| I_\alpha(f)(x) - \fint_{B(x,t)} I_\alpha(f)(y) \, dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \tag{5.1}$$

where I_α is the Riesz potential operator defined by

$$\widehat{I_\alpha(f)}(\xi) = (2\pi|\xi|)^{-\alpha} \hat{f}(\xi). \tag{5.2}$$

Then, from [1] we can see the following result.

Theorem F. *Suppose that $1 < p < \infty$ and $n \geq 2$. Let T_α be as in (5.1). Then*

$$\|T_1(f)\|_p \simeq \|f\|_p, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

In [1] this was used to prove Theorem D in Section 1 when $n \geq 2$. Theorem F is generalized to the weighted L^p spaces (see [11, 21]).

We consider square functions generalizing U_α and T_α in (1.10) and (5.1). Let

$$U_\alpha(f)(x) = \left(\int_0^\infty |f(x) - \Phi_t * f(x)|^2 \, \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0, \tag{5.3}$$

with $\Phi \in \mathcal{M}^\alpha$, where we say $\Phi \in \mathcal{M}^\alpha$, $\alpha > 0$, if Φ is a bounded function on $\mathbb{R}^n$ with compact support satisfying $\int_{\mathbb{R}^n} \Phi(x) \, dx = 1$; if $\alpha \geq 1$, we further assume that

$$\int_{\mathbb{R}^n} \Phi(x) x^\gamma \, dx = 0 \quad \text{for all } \gamma \text{ with } 1 \leq |\gamma| \leq [\alpha]. \tag{5.4}$$

When $1 \leq \alpha < 2$, (5.4) is satisfied if Φ is even; in particular, we note that $\chi_0 = \chi_{B(0,1)}/|B(0,1)| \in \mathcal{M}^\alpha$ for $0 < \alpha < 2$ and if $\Phi = \chi_0$ in (5.3), we have U_α of (1.10).

We also consider

$$T_\alpha(f)(x) = \left(\int_0^\infty |I_\alpha(f)(x) - \Phi_t * I_\alpha(f)(x)|^2 \, \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \tag{5.5}$$

where $0 < \alpha < n$ and $\Phi \in \mathcal{M}^\alpha$. If we set $\Phi = \chi_0$ in (5.5), we get T_α of (5.1).

We prove the following.

Theorem 5.1. *Suppose that T_α is as in (5.5) and $0 < \alpha < n$, $1 < p < \infty$. Let $w \in A_p$. Then*

$$\|T_\alpha(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

By Theorem 5.1 we see that U_α can be used to characterize the weighted Sobolev spaces.

Let J_α be the Bessel potential operator defined as $J_\alpha(g) = K_\alpha * g$ with

$$\hat{K}_\alpha(\xi) = (1 + 4\pi^2 |\xi|^2)^{-\alpha/2}$$

(see [24]). Let $1 < p < \infty$, $\alpha > 0$ and $w \in A_p$. The weighted Sobolev space $W_w^{\alpha,p}(\mathbb{R}^n)$ is defined to be the collection of all the functions f which can be expressed as $f = J_\alpha(g)$ with $g \in L_w^p(\mathbb{R}^n)$ and its norm is defined by $\|f\|_{p,\alpha,w} = \|g\|_{p,w}$. The weighted L^p norm inequality for the Hardy-Littlewood maximal operator with A_p weights (see [10]) implies that $J_\alpha(g) \in L_w^p$ if $g \in L_w^p$, since it is known that $|J_\alpha(g)| \leq CM(g)$ (see [24, 25]). We also note that J_α is injective on L_w^p. So, the norm $\|f\|_{p,\alpha,w}$ is well-defined.

Applying Theorem 5.1, we have the following.

Corollary 5.2. *Let $1 < p < \infty$, $w \in A_p$ and $0 < \alpha < n$. Let U_α be as in (5.3). Then $f \in W_w^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L_w^p$ and $U_\alpha(f) \in L_w^p$; furthermore,*

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|U_\alpha(f)\|_{p,w}.$$

For the case $n = 1$ and $\alpha = 1$, see Remark 5.7 below. We refer to [22, 23, 25, 29] for relevant results. See [11] for characterization of the weighted Sobolev space $W_w^{1,p}$ using square functions.

Also, we consider discrete parameter versions of T_α and U_α:

$$D_\alpha(f)(x) = \left(\sum_{k=-\infty}^{\infty} |I_\alpha(f)(x) - \Phi_{2^k} * I_\alpha(f)(x)|^2 \, 2^{-2k\alpha} \right)^{1/2}, \qquad (5.6)$$

with $0 < \alpha < n$;

$$E_\alpha(f)(x) = \left(\sum_{k=-\infty}^{\infty} |f(x) - \Phi_{2^k} * f(x)|^2 \, 2^{-2k\alpha} \right)^{1/2}, \quad \alpha > 0, \qquad (5.7)$$

where $\Phi \in \mathcal{M}^\alpha$. If we put $\Phi = \chi_0$ in (5.7), we have E_α of (1.11). We have discrete parameter analogues of Theorem 5.1 and Corollary 5.2.

Theorem 5.3. *Let $0 < \alpha < n$ and $1 < p < \infty$. Let D_α be as in (5.6). Then*

$$\|D_\alpha(f)\|_{p,w} \simeq \|f\|_{p,w}, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where w is any weight in A_p.

Corollary 5.4. *Let E_α be as in (5.7). Suppose that $1 < p < \infty$, $w \in A_p$ and $0 < \alpha < n$. Then $f \in W_w^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L_w^p$ and $E_\alpha(f) \in L_w^p$; also,*

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|E_\alpha(f)\|_{p,w}.$$

A version of Theorem 5.1 for $0 < \alpha < 2$ and $n \geq 2$ is shown in [21], where Φ is assumed to be radial. Combining the arguments of [21] with Corollary 2.11, we can relax the assumption that Φ is radial.

Here we give proofs of Theorem 5.3 and Corollary 5.4; Theorem 5.1 and Corollary 5.2 can be shown similarly.

Proof of Theorem 5.3. Recall that $\widehat{L}_\alpha(\xi) = (2\pi|\xi|)^{-\alpha}$, $0 < \alpha < n$, if $L_\alpha(x) = \tau(\alpha)|x|^{\alpha-n}$ with

$$\tau(\alpha) = \frac{\Gamma\left(n/2 - \alpha/2\right)}{\pi^{n/2} 2^\alpha \Gamma\left(\alpha/2\right)}.$$

Let

$$\psi(x) = L_\alpha(x) - \Phi * L_\alpha(x).$$

Then, we have $D_\alpha(f) = \Delta_\psi(f)$, $f \in \mathcal{S}(\mathbb{R}^n)$, by homogeneity of L_α, where D_α is as in (5.6). We observe that ψ can be written as

$$\psi(x) = \int \left(L_\alpha(x) - L_\alpha(x - y) \right) \Phi(y) \, dy. \qquad (5.8)$$

Because Φ is bounded and compactly supported and L_α is locally integrable, we see that

$$\sup_{|x| \leq 1} \left| \int L_\alpha(x - y) \Phi(y) \, dy \right| \leq C$$

for some constant C. Using this inequality in the definition of ψ, we have

$$|\psi(x)| \leq C|x|^{\alpha-n} \quad \text{for } |x| \leq 1. \qquad (5.9)$$

By applying Taylor's formula and (5.4), we can easily deduce from (5.8) that

$$|\psi(x)| \le C|x|^{\alpha-n-[\alpha]-1} \quad \text{for } |x| \ge 1. \tag{5.10}$$

Taking the Fourier transform, we see that

$$\hat{\psi}(\xi) = (2\pi|\xi|)^{-\alpha}\left(1 - \hat{\Phi}(\xi)\right). \tag{5.11}$$

By (5.4) this implies $|\hat{\psi}(\xi)| \le C|\xi|^{[\alpha]+1-\alpha}$, from which the condition (1.1) follows, since $[\alpha]+1-\alpha > 0$. It is easy to see that the conditions (1), (2) and (3) of Theorem 3.1 follow from the estimates (5.9), (5.10) and (5.11). Also, obviously we have $\sup_{k\in\mathbb{Z}}|\hat{\psi}(2^k\xi)| > 0$ for all $\xi \ne 0$. (This can be easily seen by noting that $\hat{\Phi}(\xi) \to 0$ as $|\xi| \to \infty$.) Thus we can apply Corollary 3.7 to get the equivalence of the L_w^p norms claimed. □

Proof of Corollary 5.4. Riesz potentials and Bessel potentials are related as follows.

Lemma 5.5. *Let* $\alpha > 0$, $1 < p < \infty$ *and* $w \in A_p$.
 (1) *We have*
$$(2\pi|\xi|)^\alpha = \ell(\xi)(1 + 4\pi^2|\xi|^2)^{\alpha/2}$$
 with a Fourier multiplier ℓ *for* L_w^p.
 (2) *There exists a Fourier multiplier* m *for* L_w^p *such that*
$$(1 + 4\pi^2|\xi|^2)^{\alpha/2} = m(\xi) + m(\xi)(2\pi|\xi|)^\alpha.$$

To prove this we note that

$$|\partial_\xi^\gamma \ell(\xi)| \le C_\gamma|\xi|^{-|\gamma|}, \quad \xi \in \mathbb{R}^n \setminus \{0\},$$

for all multi-indices γ and similar estimates for $m(\xi)$. So, by a theorem on Fourier multipliers for L_w^p we can get the conclusion, as in the case of the estimate for $\|M_k\|_{M^p(w)}$ in the proof of Theorem 2.5. See also [23, Lemma 4].
 When $g \in L_w^p$, $w \in A_p$, $1 < p < \infty$ and $0 < \alpha < n$, we show that

$$\|E_\alpha(J_\alpha(g))\|_{p,w} + \|J_\alpha(g)\|_{p,w} \simeq \|g\|_{p,w}. \tag{5.12}$$

We first prove (5.12) for $g \in \mathcal{S}_0(\mathbb{R}^n)$. Since $E_\alpha(J_\alpha(g)) = D_\alpha(I_{-\alpha}J_\alpha(g))$ and $I_{-\alpha}J_\alpha(g) \in \mathcal{S}(\mathbb{R}^n)$, when $g \in \mathcal{S}_0(\mathbb{R}^n)$, by Theorem 5.3 we have

$$\|E_\alpha(J_\alpha(g))\|_{p,w} \simeq \|I_{-\alpha}J_\alpha(g)\|_{p,w}, \tag{5.13}$$

where $I_{-\alpha}$ is defined by (5.2) with $-\alpha$ in place of α. Part (1) of Lemma 5.5 implies that

$$\|I_{-\alpha}J_\alpha(g)\|_{p,w} \le C\|g\|_{p,w}$$

and hence

$$\|E_\alpha(J_\alpha(g))\|_{p,w} \le C\|g\|_{p,w}. \tag{5.14}$$

On the other hand, by part (2) of Lemma 5.5 and (5.13) we have

$$\|g\|_{p,w} = \|J_{-\alpha}J_\alpha(g)\|_{p,w} \le C\|J_\alpha(g)\|_{p,w} + C\|I_{-\alpha}J_\alpha(g)\|_{p,w} \tag{5.15}$$
$$\le C\|J_\alpha(g)\|_{p,w} + C\|E_\alpha(J_\alpha(g))\|_{p,w},$$

where we recall that the Bessel potential operator J_β is defined on $\mathcal{S}(\mathbb{R}^n)$ for any $\beta \in \mathbb{R}$ by $\widehat{J_\beta(f)}(\xi) = (1 + 4\pi^2 |\xi|^2)^{-\beta/2} \hat{f}(\xi)$. Also we have

$$\|J_\alpha(g)\|_{p,w} \le C \|M(g)\|_{p,w} \le C \|g\|_{p,w}. \tag{5.16}$$

Combining (5.14), (5.15) and (5.16), we have (5.12) for $g \in \mathcal{S}_0(\mathbb{R}^n)$.

Now we show that (5.12) holds for any $g \in L_w^p$. For a positive integer N, let

$$E_\alpha^{(N)}(f)(x) = \left(\sum_{k=-N}^{N} |f(x) - \Phi_{2^k} * f(x)|^2 \, 2^{-2k\alpha} \right)^{1/2}.$$

Then $E_\alpha^{(N)}(f) \le C_N M(f)$, which implies that $E_\alpha^{(N)}$ is bounded on L_w^p. We can take a sequence $\{g_k\}$ in $\mathcal{S}_0(\mathbb{R}^n)$ such that $g_k \to g$ in L_w^p and $J_\alpha(g_k) \to J_\alpha(g)$ in L_w^p as $k \to \infty$. By (5.12) for $\mathcal{S}_0(\mathbb{R}^n)$ we see that

$$\|E_\alpha^{(N)}(J_\alpha(g_k))\|_{p,w} \le C \|g_k\|_{p,w}.$$

Letting $k \to \infty$, by L_w^p boundedness and sublinearity of $E_\alpha^{(N)}$ we have

$$\|E_\alpha^{(N)}(J_\alpha(g))\|_{p,w} \le C \|g\|_{p,w}.$$

Thus, letting $N \to \infty$, we get

$$\|E_\alpha(J_\alpha(g))\|_{p,w} \le C \|g\|_{p,w}.$$

Therefore, we have

$$\lim_{k \to \infty} \|E_\alpha(J_\alpha(g)) - E_\alpha(J_\alpha(g_k))\|_{p,w} \le \lim_{k \to \infty} \|E_\alpha(J_\alpha(g - g_k))\|_{p,w}$$
$$\le C \lim_{k \to \infty} \|g - g_k\|_{p,w} = 0.$$

Consequently, letting $k \to \infty$ in the relation

$$\|E_\alpha(J_\alpha(g_k))\|_{p,w} + \|J_\alpha(g_k)\|_{p,w} \simeq \|g_k\|_{p,w},$$

which we have already proved, we can obtain (5.12) for any $g \in L_w^p$.

To complete the proof of Corollary 5.4, it thus only remains to show that $f \in W_w^{\alpha,p}(\mathbb{R}^n)$ if $f \in L_w^p$ and $E_\alpha(f) \in L_w^p$. To prove this it is convenient to note the following.

Lemma 5.6. *Suppose that $f \in L_w^p$, $w \in A_p$, $1 < p < \infty$, $g \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha > 0$. Then we have the following.*

(1) $K_\alpha * (f * g)(x) = (K_\alpha * f) * g(x) = (K_\alpha * g) * f(x)$ *for every $x \in \mathbb{R}^n$;*
(2) $\int_{\mathbb{R}^n} (K_\alpha * f)(y) g(y) \, dy = \int_{\mathbb{R}^n} (K_\alpha * g)(y) f(y) \, dy.$

Proof. To prove part (1), by Fubini's theorem it suffices to show that

$$I = \iint K_\alpha(x - z - y)|f(y)||g(z)| \, dy \, dz < \infty.$$

This is obvious, for

$$I \leq C \int M(f)(x-z)|g(z)| \, dz = C \int M(f)(z)|g(x-z)| \, dz$$

$$\leq C\|M(f)\|_{p,w} \left(\int |g(x-z)|^{p'} w(z)^{-p'/p} \, dz \right)^{1/p'}$$

$$\leq C\|f\|_{p,w} \left(\int |g(x-z)|^{p'} w(z)^{-p'/p} \, dz \right)^{1/p'},$$

where the last integral is finite since $g \in \mathcal{S}(\mathbb{R}^n)$, $w^{-p'/p} \in A_{p'}$ and $\mathcal{S}(\mathbb{R}^n) \subset L_v^r$ for $v \in A_r$, $1 < r < \infty$ (see [10, p. 412] for a related result).

Part (2) follows from part (1) by putting $x = 0$ since K_α is radial. $\square$

Let $f \in L_w^p$ and $E_\alpha(f) \in L_w^p$. We take $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int \varphi(x) \, dx = 1$. Let $f^{(\epsilon)}(x) = \varphi_\epsilon * f(x)$ and $g^{(\epsilon)}(x) = J_{-\alpha}(\varphi_\epsilon) * f(x)$. Then, note that $g^{(\epsilon)} \in L_w^p$ and $f^{(\epsilon)} = J_\alpha(g^{(\epsilon)})$ by part (1) of Lemma 5.6.

By (5.12) we have

$$\|E_\alpha(f^{(\epsilon)})\|_{p,w} + \|f^{(\epsilon)}\|_{p,w} \simeq \|g^{(\epsilon)}\|_{p,w}. \tag{5.17}$$

We note that

$$\sup_{\epsilon > 0} \|f^{(\epsilon)}\|_{p,w} \leq C\|M(f)\|_{p,w} \leq C\|f\|_{p,w}. \tag{5.18}$$

Also, Minkowski's inequality implies that

$$E_\alpha(f^{(\epsilon)})(x) = \left(\sum_{k=-\infty}^{\infty} |\varphi_\epsilon * f(x) - \Phi_{2^k} * \varphi_\epsilon * f(x)|^2 \, 2^{-2k\alpha} \right)^{1/2}$$

$$\leq \int_{\mathbb{R}^n} |\varphi_\epsilon(y)| \left(\sum_{k=-\infty}^{\infty} |f(x-y) - \Phi_{2^k} * f(x-y)|^2 \, 2^{-2k\alpha} \right)^{1/2} \, dy$$

$$\leq CM(E_\alpha(f))(x).$$

Thus

$$\sup_{\epsilon > 0} \|E_\alpha(f^{(\epsilon)})\|_{p,w} \leq C\|M(E_\alpha(f))\|_{p,w} \leq C\|E_\alpha(f)\|_{p,w},$$

which combined with (5.17) and (5.18) implies that $\sup_{\epsilon>0} \|g^{(\epsilon)}\|_{p,w} < \infty$.

Therefore we can choose a sequence $\{g^{(\epsilon_k)}\}$, $\epsilon_k \to 0$, which converges weakly in L_w^p. Let $g^{(\epsilon_k)} \to g$ weakly in L_w^p. Then, since $\{f^{(\epsilon_k)}\}$ converges to f in L_w^p, we can conclude that $f = J_\alpha(g)$. To show this, let $\Lambda_h(f) = \int f(x)h(x) \, dx$ for $h \in \mathcal{S}(\mathbb{R}^n)$. Then it is easy to see that Λ_h is a bounded linear functional on L_w^p for every $h \in \mathcal{S}(\mathbb{R}^n)$, since $|\Lambda_h(f)| \leq \|f\|_{p,w}\|h\|_{p',w^{-p'/p}}$ by Hölder's inequality. Thus, for any $h \in \mathcal{S}(\mathbb{R}^n)$, applying part (2) of Lemma 5.6

and noting $J_\alpha(h) \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\int f(x)h(x)\,dx = \lim_k \int f^{(\epsilon_k)}(x)h(x)\,dx = \lim_k \int J_\alpha(g^{(\epsilon_k)})(x)h(x)\,dx$$

$$= \lim_k \int g^{(\epsilon_k)}(x)J_\alpha(h)(x)\,dx = \int g(x)J_\alpha(h)(x)\,dx$$

$$= \int J_\alpha(g)(x)h(x)\,dx.$$

This implies that $f = J_\alpha(g)$ and hence $f \in W_w^{\alpha,p}(\mathbb{R}^n)$. This completes the proof of Corollary 5.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.7. Let $\psi = \mathrm{sgn} - \mathrm{sgn} * \Phi$ on $\mathbb{R}$, where $\Phi \in \mathcal{M}^1$. We note that $\hat{\psi}(\xi) = -i\pi^{-1}\xi^{-1}(1 - \hat{\Phi}(\xi))$. We have results analogous to Theorems 5.1 and 5.3 for g_ψ and Δ_ψ, respectively, with similar proofs. They can be applied to prove results generalizing Corollaries 5.2 and 5.4 to the case $n = 1$ and $\alpha = 1$ by arguments similar to those used for the corollaries.

6. Proof of (4.7)

In this section we give a proof of the estimate (4.7) for completeness. Put $\psi = \varphi^{(\alpha)}$. To prove (4.7), assuming $|y| < |x|/2$, we write

$$L = \int_0^\infty |t^{-1}\psi((x-y)/t) - t^{-1}\psi(x/t)|^2 \, \frac{dt}{t}.$$

We first assume $x > 0$ and $y > 0$. By the change of variables $x/t = u$ we have

$$L = x^{-2} \int_0^\infty |\psi(u - uy/x) - \psi(u)|^2 u \, du = I + II,$$

where

$$I = x^{-2} \int_0^1 |\psi(u - uy/x) - \psi(u)|^2 u \, du,$$

$$II = x^{-2} \int_1^\infty |\psi(u - uy/x) - \psi(u)|^2 u \, du.$$

We estimate I and II separately. We see that

$$II = x^{-2} \int_1^\infty |\psi(u - uy/x)|^2 u \, du$$

$$= \alpha^2 x^{-2} \int_1^{x/(x-y)} (1 - |u(1 - y/x)|)^{2(\alpha-1)} u \, du.$$

Thus, by the change of variables $w = u(x - y)/x$, we have

$$II = \alpha^2 (x - y)^{-2} \int_{(x-y)/x}^1 (1 - w)^{2(\alpha-1)} w \, dw$$

$$\leq \alpha^2 (x - y)^{-2} \int_{(x-y)/x}^1 (1 - w)^{2(\alpha-1)} \, dw,$$

which implies that

$$II \leq \alpha^2 (x-y)^{-2} (2\alpha - 1)^{-1} (y/x)^{2\alpha - 1} \leq C_\alpha y^{2\alpha - 1} x^{-1-2\alpha}. \qquad (6.1)$$

To deal with I, we write

$$I = \alpha^2 x^{-2} \int_0^1 |(1 - u(1 - y/x))^{\alpha - 1} - (1 - u)^{\alpha - 1}|^2 u \, du = I_1 + I_2,$$

where

$$I_1 = \alpha^2 x^{-2} \int_0^{1 - 2y/x} |(1 - u(1 - y/x))^{\alpha - 1} - (1 - u)^{\alpha - 1}|^2 u \, du,$$

$$I_2 = \alpha^2 x^{-2} \int_{1 - 2y/x}^1 |(1 - u(1 - y/x))^{\alpha - 1} - (1 - u)^{\alpha - 1}|^2 u \, du.$$

We observe that

$$\int_{1 - 2y/x}^1 (1 - u(1 - y/x))^{2(\alpha - 1)} \, du \qquad (6.2)$$

$$= x(x - y)^{-1} \int_{(1 - 2y/x)(x - y)/x}^{(x - y)/x} (1 - w)^{2(\alpha - 1)} \, dw$$

$$\leq C_\alpha \left((1 - (1 - 2y/x)(x - y)/x)^{2\alpha - 1} - (1 - (x - y)/x)^{2\alpha - 1} \right)$$

$$\leq C_\alpha (y/x)^{2\alpha - 1}.$$

Also, we have

$$\int_{1 - 2y/x}^1 (1 - u)^{2(\alpha - 1)} \, du \leq C_\alpha (y/x)^{2\alpha - 1}. \qquad (6.3)$$

By (6.2) and (6.3) we see that

$$I_2 \leq C_\alpha x^{-2} (y/x)^{2\alpha - 1} = C_\alpha y^{2\alpha - 1} x^{-1-2\alpha}. \qquad (6.4)$$

To estimate I_1 we recall that $1/2 < \alpha < 3/2$. By the mean value theorem, we have

$$I_1 \leq C x^{-2} (y/x)^2 \int_0^{1 - 2y/x} (1 - u)^{2(\alpha - 2)} \, du \qquad (6.5)$$

$$\leq C x^{-2} (y/x)^2 (2y/x)^{2\alpha - 3} = C y^{2\alpha - 1} x^{-2\alpha - 1}.$$

The estimate $I \leq C_\alpha y^{2\alpha - 1} x^{-1-2\alpha}$ follows from (6.4) and (6.5), which combined with (6.1) implies

$$L \leq C_\alpha y^{2\alpha - 1} x^{-1-2\alpha},$$

when $x > 0$, $y > 0$.

Next we deal with the case $x > 0$, $y < 0$. In this case we also consider the analogous decomposition $L = I + II$. Since ψ is supported in $[-1, 1]$ and

$x > 0, y < 0$, we see that $II = 0$. Also, $I = I_1 + I_2$, where

$$I_1 = \alpha^2 x^{-2} \int_0^{1-2|y|/x} |(1 - u(1 - y/x))^{\alpha-1} - (1 - u)^{\alpha-1}|^2 u\, du,$$

$$I_2 = x^{-2} \int_{1-2|y|/x}^1 |\psi(u(1 - y/x)) - \psi(u)|^2 u\, du.$$

To estimate I_2, we see that

$$\int_{1-2|y|/x}^1 |\psi(u(1-y/x))|^2 u\, du \leq \alpha^2 \int_{1-2|y|/x}^{x/(x-y)} |1 - u(1-y/x)|^{2(\alpha-1)}\, du$$

$$= \alpha^2 x(x-y)^{-1} \int_{(x-y)(x+2y)/x^2}^1 (1-w)^{2(\alpha-1)}\, dw$$

$$= \alpha^2 x(x-y)^{-1}(2\alpha - 1)^{-1}(|y|/x + 2(y/x)^2)^{2\alpha-1}$$

$$\leq C_\alpha |y/x|^{2\alpha-1}.$$

Similarly,

$$\int_{1-2|y|/x}^1 (1-u)^{2(\alpha-1)} u\, du \leq C_\alpha |y/x|^{2\alpha-1}.$$

Thus

$$I_2 \leq C|y|^{2\alpha-1} x^{-2\alpha-1}. \tag{6.6}$$

On the other hand, by the mean value theorem,

$$I_1 \leq \alpha^2 x^{-2} \int_0^{1-2|y|/x} \left(|y|x^{-1}|\alpha - 1||(1 - u(1 - y/x))|^{\alpha-2}\right)^2\, du \tag{6.7}$$

$$\leq Cy^2 x^{-4} x(x-y)^{-1} \int_0^{(x+2y)(x-y)/x^2} (1-u)^{2(\alpha-2)}\, du$$

$$= Cy^2 x^{-3}(x-y)^{-1}(3 - 2\alpha)^{-1}\left((|y|/x + 2(y/x)^2)^{2\alpha-3} - 1\right)$$

$$\leq C_\alpha |y|^{2\alpha-1} x^{-2\alpha-1}.$$

The estimates (6.6) and (6.7) imply that $I \leq C|y|^{2\alpha-1} x^{-2\alpha-1}$ for $x > 0, y < 0$.

Since ψ is odd, we observe that

$$L = \int_0^\infty |t^{-1}\psi((-x+y)/t) - t^{-1}\psi(-x/t)|^2\, \frac{dt}{t}.$$

Thus, the results for the cases $x < 0$, $y > 0$ and $x < 0$, $y < 0$ will follow from the results for the cases $x > 0$, $y < 0$ and $x > 0$, $y > 0$, respectively.

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