Spherical square functions of Marcinkiewicz type with Riesz potentials

SPHERICAL SQUARE FUNCTIONS OF MARCINKIEWICZ TYPE WITH RIESZ POTENTIALS

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Abstract. We prove a pointwise equivalence between a spherical square function composed with the Riesz potential and a Littlewood-Paley function arising from the Bochner-Riesz operators. Also, its application to the theory of Sobolev spaces will be given.

1. Introduction

Let

$$\nu(f)(x) = \left(\int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Then $\mu(f) = \nu(\mathcal{I}(f))$ is the function of Marcinkiewicz, where $\mathcal{I}(f)(x) = \int_0^x f(y)\,dy$. If $f \in L^p(\mathbb{R})$, $1 < p < \infty$, we have

$$\|\mu(f)\|_p \simeq \|f\|_p, \tag{1.1}$$

which means that there exist positive constants c_1, c_2 independent of f such that

$$c_1\|f\|_p \le \|\mu(f)\|_p \le c_2\|f\|_p.$$

In other words, we have $\|\nu(f)\|_p \simeq \|f'\|_p$ if f is in the Sobolev space $W^{1,p}(\mathbb{R})$, $1 < p < \infty$.

The Marcinkiewicz function was introduced by J. Marcinkiewicz [5] in 1938 in the setting of periodic functions on the torus $\mathbb{T}$. It can be used to investigate properties of functions such as differentiability and finiteness of norms in function spaces. Zygmund [10] proved a periodic analogue of (1.1), which was conjectured in [5]. The non-periodic version (1.1) was proved by Waterman [8].

The Marcinkiewicz function is a kind of Littlewood-Paley functions; $\mu(f)$ can be realized as

$$\mu(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-1}\psi(t^{-1}x)$ with $\psi(x) = \chi_{[0,1]}(x) - \chi_{[-1,0]}(x)$; here χ_E denotes the characteristic function of a set E. We observe that

$$f(x+t) + f(x-t) - 2f(x) = \int_{S^0} (f(x-t\theta) - f(x))\,d\sigma(\theta),$$

where $d\sigma$ is a uniform measure on $S^0 = \{-1, 1\}$ such that $\sigma(\{-1\}) = 1$, $\sigma(\{1\}) = 1$.

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In this note we assume that $n \geq 2$ and consider the square function

$$D^\alpha(f)(x) = \left(\int_0^\infty \left| t^{-\alpha} \int_{S^{n-1}} (f(x - t\theta) - f(x))\, d\sigma(\theta) \right|^2 \frac{dt}{t} \right)^{1/2}, \tag{1.2}$$

for appropriate functions f, where $d\sigma$ is the Lebesgue surface measure on S^{n-1}. Then $D^\alpha(f)$ with $\alpha = 1$ can be regarded as a generalization to higher dimensions of $\nu(f)$. We shall see that $D^\alpha(f)$ also can be used to characterize Sobolev norms. This will be accomplished through a relation between $D^\alpha(f)$ and another square function arising from the Bochner-Riesz operators. Let

$$S_R^\beta(f)(x) = \int_{|\xi|<R} \hat{f}(\xi)(1 - R^{-2}|\xi|^2)^\beta\, e^{2\pi i\langle x,\xi\rangle}\, d\xi = H_{R^{-1}}^\beta * f(x)$$

be the Bochner-Riesz mean of order β on $\mathbb{R}^n$, where

$$H_{R^{-1}}^\beta(x) = R^n H^\beta(Rx), \quad H^\beta(x) = \pi^{-\beta}\Gamma(\beta+1)|x|^{-(n/2+\beta)} J_{n/2+\beta}(2\pi|x|)$$

with J_ν denoting the Bessel function of the first kind of order ν and $\hat{f}$ is the Fourier transform defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i\langle x,\xi\rangle}\, dx, \quad \langle x, \xi\rangle = \sum_{k=1}^n x_k \xi_k.$$

We recall a Littlewood-Paley operator σ_β, $\mathrm{Re}(\beta) > 0$, defined from the Bochner-Riesz means as

$$\begin{aligned} \sigma_\beta(f)(x) &= \left(\int_0^\infty \left| R\partial_R S_R^\beta(f)(x) \right|^2 \frac{dR}{R} \right)^{1/2} \\ &= \left(\int_0^\infty \left| -2\beta \left(S_R^\beta(f)(x) - S_R^{\beta-1}(f)(x) \right) \right|^2 \frac{dR}{R} \right)^{1/2}, \end{aligned} \tag{1.3}$$

where $\mathrm{Re}(\beta)$ denotes the real part of the complex number β and $\partial_R = \partial/\partial_R$. Also, let I_α be the Riesz potential operator defined as

$$\widehat{I_\alpha(f)}(\xi) = |\xi|^{-\alpha}\hat{f}(\xi). \tag{1.4}$$

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$. Let $\mathcal{S}_0(\mathbb{R}^n)$ be the subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of functions f with $\hat{f}$ vanishing in a neighborhood of the origin. We shall prove the following.

Theorem 1.1. *Let $0 < \alpha < 2$. Then if $\beta = \alpha + \frac{n}{2}$, we have*

$$\sigma_\beta(f)(x) \approx D^\alpha(I_\alpha f)(x)$$

for $f \in \mathcal{S}_0(\mathbb{R}^n)$, where D^α, I_α and σ_β are as in (1.2), (1.4) *and* (1.3)*, respectively.*

Here $\sigma_\beta(f)(x) \approx D^\alpha(I_\alpha f)(x)$ means that there exist positive constants A, B independent of f and x such that

$$A\sigma_\beta(f)(x) \leq D^\alpha(I_\alpha f)(x) \leq B\sigma_\beta(f)(x).$$

A version of Theorem 1.1 was shown in [4] for the range $0 < \alpha < 1$. In this note we shall extend this range of α to $0 < \alpha < 2$. The difference from [4] that enables us to improve the range of α mainly comes from the estimate in part (1) of Lemma 2.5.

In Section 2, we shall give an almost self-contained proof of Theorem 1.1 except that Lemmas 2.2, 2.3 from [9] and the formula (2.9) are taken for granted. In

Section 3, applications of Theoerm 1.1 to the theory of Sobolev spaces will be given.

2. Proof of Theorem 1.1

For a fixed function $f \in \mathcal{S}_0(\mathbb{R}^n)$ and a fixed point $x \in \mathbb{R}^n$, let

$$\varphi(t) = \varphi(t; x, f) = \int_{S^{n-1}} f(x - ty')\, d\sigma(y'), \tag{2.1}$$

$$\theta(t) = \theta(t; x, f) = t\frac{\partial}{\partial t}\varphi(t; x, f) = -\int_{S^{n-1}} \langle ty', \nabla f(x - ty')\rangle\, d\sigma(y'), \tag{2.2}$$

where $\nabla f(x) = (\partial_1 f(x), \dots, \partial_n f(x))$, $\partial_j = \partial/\partial x_j$.

Let $\operatorname{Re}\alpha > -n$. Define

$$\widehat{I^\alpha(f)}(\xi) = |\xi|^\alpha \hat{f}(\xi), \tag{2.3}$$

and for $\operatorname{Re}\beta > -1$ consider

$$S_R^\beta(I^\alpha f)(x) = \int_{|\xi|<R} \widehat{f}(\xi)|\xi|^\alpha(1 - R^{-2}|\xi|^2)^\beta\ e^{2\pi i\langle x,\xi\rangle}\, d\xi = R^\alpha L^{\alpha,\beta}_{R^{-1}} * f(x), \tag{2.4}$$

where

$$\begin{aligned} L^{\alpha,\beta}(x) &= \int_{|\xi|<1} |\xi|^\alpha(1 - |\xi|^2)^\beta\ e^{2\pi i\langle x,\xi\rangle}\, d\xi \\ &= \int_0^1 r^{n+\alpha}(1 - r^2)^\beta 2\pi(r|x|)^{-(n-2)/2} J_{(n-2)/2}(2\pi r|x|)\, \frac{dr}{r} \\ &= (2\pi)^{n/2}\int_0^1 r^{n+\alpha}(1 - r^2)^\beta V_{(n-2)/2}(2\pi r|x|)\, \frac{dr}{r}, \end{aligned}$$

with

$$V_\nu(r) = r^{-\nu}J_\nu(r).$$

We write the formula in (2.4) by using φ in (2.1).

Lemma 2.1. *Suppose that* $\operatorname{Re}\alpha > -n$, $\operatorname{Re}\beta > -1$. *Let*

$$l_{\alpha,\beta}(s) = (2\pi)^{n/2}\int_0^1 r^{n+\alpha}(1 - r^2)^\beta V_{(n-2)/2}(2\pi rs)\, \frac{dr}{r}.$$

Then we have

$$S_R^\beta(I^\alpha f)(x) = R^{n+\alpha}\int_0^\infty l_{\alpha,\beta}(Rs)\varphi(s; x, f)s^n\, \frac{ds}{s}.$$

Proof. By (2.4) we have

$$S_R^\beta(I^\alpha f)(x) = R^\alpha L^{\alpha,\beta}_{R^{-1}} * f(x) = R^\alpha\int_{\mathbb{R}^n} f(x - y)L^{\alpha,\beta}_{R^{-1}}(y)\, dy.$$

Using polar coordinates and recalling the definition of $L^{\alpha,\beta}$ in the last integral, by (2.1) we reach the conclusion.

□

We use the following formulas.

Lemma 2.2. *If* $\mathrm{Re}(\mu) > -1$, $\mathrm{Re}(\nu) > -1$, $t > 0$,

$$J_{\mu+\nu+1}(t) = \frac{t^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^1 J_\mu(ts) s^{\mu+1} (1-s^2)^\nu \, ds.$$

This can be rewritten as

$$V_{\mu+\nu+1}(t) = \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^1 V_\mu(ts) s^{2\mu+1} (1-s^2)^\nu \, ds.$$

Lemma 2.3. *If* $0 < \mathrm{Re}(\mu) < \mathrm{Re}(\nu) + 1/2$,

$$\int_0^\infty t^{\mu-1} V_\nu(t)\, dt = \frac{\Gamma(\mu/2)}{2^{\nu-\mu+1}\Gamma(\nu-\mu/2+1)}.$$

See [9, p.373] and [9, p.391] for Lemma 2.2 and Lemma 2.3, respectively.

We need the expression of $\sigma_\beta(I^\alpha f)$ in (2.7) below. To obtain it we show the following.

Lemma 2.4. *Let* $\mathrm{Re}\,\alpha > -n$, $\mathrm{Re}\,\beta > 0$. *Then*

$$\begin{aligned} &R\partial_R S_R^\beta(I^\alpha f)(x) \\ &\quad = -2\beta(2\pi)^{n/2} R^\alpha \int_0^\infty \theta(s/R) s^{n-1}\, ds \int_0^1 r^{n+\alpha+2}(1-r^2)^{\beta-1} V_{n/2}(2\pi rs)\, \frac{dr}{r}. \end{aligned}$$

Proof. We first compute $R\partial_R(R^{n+\alpha} l_\beta(Rs))$ as follows.

$$\begin{aligned} R\partial_R(R^{n+\alpha} l_{\alpha,\beta}(Rs)) &= R\partial_R\left[(2\pi)^{n/2} R^{n+\alpha} \int_0^1 r^{n+\alpha}(1-r^2)^\beta V_{(n-2)/2}(2\pi Rrs)\, \frac{dr}{r}\right] \\ &= R\partial_R\left[(2\pi)^{n/2} \int_0^R r^{n+\alpha}\left(1-\frac{r^2}{R^2}\right)^\beta V_{(n-2)/2}(2\pi rs)\, \frac{dr}{r}\right] \\ &= R(2\pi)^{n/2} \int_0^R r^{n+\alpha}\beta\left(1-\frac{r^2}{R^2}\right)^{\beta-1} (2r^2R^{-3}) V_{(n-2)/2}(2\pi rs)\, \frac{dr}{r} \\ &= 2\beta(2\pi)^{n/2} R^{n+\alpha} \int_0^1 r^{n+\alpha}\left(1-r^2\right)^{\beta-1} r^2 V_{(n-2)/2}(2\pi Rrs)\, \frac{dr}{r}. \end{aligned}$$

Therefore, from Lemma 2.1 it follows that

$$\begin{aligned} &(2.5)\quad R\partial_R S_R^\beta(I^\alpha f)(x) \\ &= 2\beta(2\pi)^{n/2} R^{n+\alpha} \int_0^\infty \varphi(s;x,f) s^{n-1}\, ds \int_0^1 r^{n+\alpha}\left(1-r^2\right)^{\beta-1} r^2 V_{(n-2)/2}(2\pi Rrs)\, \frac{dr}{r}. \end{aligned}$$

By Lemma 2.2 we have

$$\begin{aligned} \int_0^u V_{(n-2)/2}(2\pi Rrs) s^{n-1}\, ds &= u^n \int_0^1 V_{(n-2)/2}(2\pi Rrsu) s^{n-1}\, ds \\ &= u^n \Gamma(1) V_{n/2}(2\pi rRu) = u^n V_{n/2}(2\pi rRu). \end{aligned}$$

Thus applying integration by parts in (2.5), we see that

$$\begin{aligned} &R\partial_R S_R^\beta(I^\alpha f)(x) \\ &= -2\beta(2\pi)^{n/2}R^{n+\alpha}\int_0^\infty \varphi'(s)\,ds\int_0^1 r^{n+\alpha}\left(1-r^2\right)^{\beta-1} r^2 s^n V_{n/2}(2\pi Rrs)\,\frac{dr}{r} \\ &= -2\beta(2\pi)^{n/2}R^{n+\alpha}\int_0^\infty \theta(s)s^{n-1}\,ds\int_0^1 r^{n+\alpha}\left(1-r^2\right)^{\beta-1} r^2 V_{n/2}(2\pi Rrs)\,\frac{dr}{r} \\ &= -2\beta(2\pi)^{n/2}R^{\alpha}\int_0^\infty \theta(s/R)s^{n-1}\,ds\int_0^1 r^{n+\alpha+2}\left(1-r^2\right)^{\beta-1} V_{n/2}(2\pi rs)\,\frac{dr}{r}, \end{aligned}$$

where $\varphi(s)=\varphi(s;x,f)$ and $\theta(s)=\theta(s;x,f)$ are as in (2.1) and (2.2), respectively. This completes the proof. □

For appropriate complex numbers α, β, let

$$\Phi_{\alpha,\beta}(s) = s^{\alpha+n}\int_0^1 r^{n+\alpha+1}\left(1-r^2\right)^{\beta-1} V_{n/2}(2\pi rs)\,dr \tag{2.6}$$

and

$$\theta_\alpha(t)=t^{-\alpha}\theta(t)=t^{-\alpha}\theta(t;x,f).$$

Then Lemma 2.4 implies that

$$R\partial_R S_R^\beta(I^\alpha f)(x) = -2\beta(2\pi)^{n/2}\int_0^\infty \Phi_{\alpha,\beta}(s)\theta_\alpha(sR^{-1})\,\frac{ds}{s}$$

for $\operatorname{Re}\alpha > -n$, $\operatorname{Re}\beta>0$. Define

$$K_{\alpha,\beta}(u) = -2\beta(2\pi)^{n/2}\Phi_{\alpha,\beta}(e^u),$$

$$\Theta_\alpha(u)=\Theta_\alpha(u,x,f)=\theta_\alpha(e^{-u}),\quad \Theta(u)=\theta(e^{-u}).$$

Then by change of variables $s=e^v$, $R=e^u$ we have

$$R\partial_R S_R^\beta(I^\alpha f)(x) = \int_{-\infty}^\infty K_{\alpha,\beta}(v)\Theta_\alpha(u-v)\,dv,$$

and hence by (1.3),

$$\sigma_\beta(I^\alpha f)(x)^2 = \int_{-\infty}^\infty |K_{\alpha,\beta}*\Theta_\alpha(u)|^2\,du. \tag{2.7}$$

In proving Theorem 1.1 we need Proposition 2.10 below. To show it we first state some properties of Θ_α and $K_{\alpha,\beta}$ (Lemmas 2.5 and 2.6).

Lemma 2.5. *We have the following estimates for Θ_α, $\alpha\in\mathbb{C}$.*

(1) $|\Theta_\alpha(u)| \le Ce^{u\operatorname{Re}(\alpha)}e^{-2u}$ *for* $u\ge 0$*;*
(2) *if* $u<0$, $|\Theta_\alpha(u)|\le C_m e^{u\operatorname{Re}(\alpha)}e^{mu}$ *for any* $m>0$.

Further, we have similar estimates for the derivatives $(d/du)^k\Theta_\alpha$, $k=1,2,\dots$. In particular, $\Theta_\alpha\in\mathcal{S}(\mathbb{R})$ if $\operatorname{Re}(\alpha)<2$.

Proof. Recall that $\theta(t) = -\int_{S^{n-1}}\langle ty',\nabla f(x-ty')\rangle\,d\sigma(y')$ and $\Theta_\alpha(u)=e^{u\alpha}\theta(e^{-u})$. Thus, part (2) follows easily since $|\nabla f(x)|\le C_m(1+|x|)^{-m-1}$ and $|x-e^{-u}y'|\ge e^{-u}-|x|\ge e^{-u}/2$ if $e^{-u}\ge 2|x|$.

To prove part (1), we note that

$$\theta(t) = -\int_{S^{n-1}}\langle ty',\nabla f(x-ty')-\nabla f(x)\rangle\,d\sigma(y'),$$

since $\int_{S^{n-1}} \langle ty', w\rangle \, d\sigma(y') = 0$ for any $w \in \mathbb{R}^n$. So $\theta(t) = O(t^2)$ as $t \to 0$, which proves part (1).

By a direct computation, we can prove the result for the derivatives $(d/du)^k\Theta_\alpha$ similarly. □

Lemma 2.6. *The following results hold for* $K_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{C}$.

(1) *If* $\mathrm{Re}(\alpha) > -n-2$ *and* $\mathrm{Re}(\beta) > 0$,

$$|K_{\alpha,\beta}(u)| \le C_{\alpha,\beta} e^{(n+\mathrm{Re}(\alpha))u}, \quad u \in \mathbb{R}.$$

(2) *If* $\mathrm{Re}(\alpha) > -n/2-2$ *and* $\mathrm{Re}(\beta) > 0$,

$$|K_{\alpha,\beta}(u)| \le C_{\alpha,\beta} e^{(n/2+\mathrm{Re}(\alpha))u}, \quad u \in \mathbb{R}.$$

(3) *If* $-n/2 > \mathrm{Re}(\alpha) > -n/2-1$ *and* $\mathrm{Re}(\beta) > 0$, *then*

$$|K_{\alpha,\beta}(u)| \le C_{\alpha,\beta} e^{-\delta|u|}, \quad u \in \mathbb{R},$$

where $\delta = \min(n + \mathrm{Re}(\alpha), -\mathrm{Re}(\alpha) - n/2) > 0$.

Proof. Since $V_{n/2}$ is bounded, by (2.6) we have part (1), where we assume that $\mathrm{Re}(\alpha) > -n-2$ and $\mathrm{Re}(\beta) > 0$ for the integrabilities on $[0,1]$ of $r^{n+\alpha+1}$ and $\left(1-r^2\right)^{\beta-1}$, respectively. By (2.6), we also have

$$\Phi_{\alpha,\beta}(s) = (2\pi)^{-n/2} s^{\alpha+n/2} \int_0^1 r^{n/2+\alpha+1} \left(1-r^2\right)^{\beta-1} J_{n/2}(2\pi r s)\, dr.$$

This implies part (2). Part (3) follows from the estimates of (1) and (2). □

Let

$$G(\alpha,\beta) = \int_{-\infty}^{\infty} K_{\alpha,\beta} * \Theta_\alpha(u) g(u)\, du, \tag{2.8}$$

where $g \in C_0^\infty(\mathbb{R})$. By Lemmas 2.5 and 2.6, the convolution $K_{\alpha,\beta} * \Theta_\alpha$ can be defined and $G(\alpha,\beta)$ is analytic in α and β if $\mathrm{Re}(\alpha) > -n-2$, $\mathrm{Re}(\beta) > 0$. Also, if $-n/2 > \mathrm{Re}(\alpha) > -n/2-1$ and $\mathrm{Re}(\beta) > 0$, then $K_{\alpha,\beta}, \Theta_\alpha \in L^1(\mathbb{R})$; in this case we have

$$G(\alpha,\beta) = \int_{-\infty}^{\infty} \widehat{K}_{\alpha,\beta}(\xi)\widehat{\Theta}_\alpha(\xi)\hat{g}(-\xi)\, d\xi.$$

An explicit form of the Fourier transform of $K_{\alpha,\beta}$ needed is stated in the next result.

Lemma 2.7. *If* $-n/2-1 < \mathrm{Re}(\alpha) < -n/2$ *and* $\mathrm{Re}(\beta) > 0$, *then* $\widehat{K}_{\alpha,\beta}(\xi) = \Psi_{\alpha,\beta}(\xi)$, *where*

$$\Psi_{\alpha,\beta}(\xi) = -2^{-1}\beta\pi^{-n/2-\alpha+2\pi i\xi} \frac{\Gamma((n+\alpha-2\pi i\xi)/2)\Gamma(\pi i\xi+1)\Gamma(\beta)}{\Gamma(-\alpha/2+\pi i\xi+1)\Gamma(\pi i\xi+1+\beta)}.$$

Proof. Let $-\operatorname{Re}(\alpha)-n<\operatorname{Re}(\zeta)<-\operatorname{Re}(\alpha)-n/2$ and $-n/2-1<\operatorname{Re}(\alpha)<-n/2$, $\operatorname{Re}(\beta)>0$. Then by Fubini's theorem we have

$$\begin{aligned}
\widehat{K}_{\alpha,\beta}\left(\frac{\zeta}{-2\pi i}\right) &= \int_{-\infty}^{\infty} e^{\zeta u} K_{\alpha,\beta}(u)\,du \\
&= -2\beta(2\pi)^{n/2}\int_0^\infty t^{n+\alpha+\zeta-1}\left(\int_0^1 r^{n+\alpha+1}\left(1-r^2\right)^{\beta-1} V_{n/2}(2\pi rt)\,dr\right)dt \\
&= -2\beta(2\pi)^{n/2}\int_0^1 r^{n+\alpha+1}\left(1-r^2\right)^{\beta-1}\left(\int_0^\infty t^{n+\alpha+\zeta-1}V_{n/2}(2\pi rt)\,dt\right)dr.
\end{aligned}$$

Using Lemma 2.3, we see that

$$\begin{aligned}
\int_0^\infty t^{n+\alpha+\zeta-1}V_{n/2}(2\pi rt)\,dt &= (2\pi r)^{-n-\alpha-\zeta}\int_0^\infty t^{n+\alpha+\zeta-1}V_{n/2}(t)\,dt \\
&= (2\pi)^{-n-\alpha-\zeta}2^{n/2+\alpha+\zeta-1}\frac{\Gamma((n+\alpha+\zeta)/2)}{\Gamma(-\alpha/2-\zeta/2+1)}r^{-n-\alpha-\zeta}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\widehat{K}_{\alpha,\beta}\left(\frac{\zeta}{-2\pi i}\right) \\
&= -2\beta(2\pi)^{-n/2-\alpha-\zeta}2^{n/2+\alpha+\zeta-1}\frac{\Gamma\left((n+\alpha+\zeta)/2\right)}{\Gamma(-\alpha/2-\zeta/2+1)}\int_0^1 r^{1-\zeta}(1-r^2)^{\beta-1}\,dr \\
&= -\beta\pi^{-n/2-\alpha-\zeta}\frac{\Gamma\left((n+\alpha+\zeta)/2\right)}{\Gamma(-\alpha/2-\zeta/2+1)}2^{-1}\int_0^1 t^{-\zeta/2}(1-t)^{\beta-1}\,dt \\
&= -2^{-1}\beta\pi^{-n/2-\alpha-\zeta}\frac{\Gamma\left((n+\alpha+\zeta)/2\right)}{\Gamma(-\alpha/2-\zeta/2+1)}\frac{\Gamma(-\zeta/2+1)\Gamma(\beta)}{\Gamma(-\zeta/2+1+\beta)}.
\end{aligned}$$

Putting $\zeta=-2\pi i\xi$, $\xi\in\mathbb{R}$, we reach the conclusion. □

In proving Proposition 2.10 we also need the following.

Lemma 2.8. *Let* $\Psi_{\alpha,\beta}$ *be as in Lemma* 2.7 *and*

$$H(\alpha,\beta)=\int_{-\infty}^{\infty}\Psi_{\alpha,\beta}(\xi)\widehat{\Theta}_\alpha(\xi)\hat{g}(-\xi)\,d\xi.$$

Then $H(\alpha,\beta)$ *is analytic in* α *and* β *if* $-n<\operatorname{Re}(\alpha)<2$, $\operatorname{Re}(\beta)>0$.

To prove this we apply the following.

Lemma 2.9 (asymptotic formula for the gamma function)**.** *Let* a, $\xi\in\mathbb{R}$, $a>0$. *Then we have*

$$\lim_{|\xi|\to\infty}\frac{|\Gamma(a+i\xi)|}{\sqrt{2\pi}e^{-\pi|\xi|/2}|\xi|^{a-1/2}}=1.$$

This is well-known. In Section 4, we shall give a proof for completeness based on the formula

$$\lim_{\operatorname{Re}(z)\geq c>0,|z|\to\infty}\frac{\Gamma(z)}{\sqrt{2\pi}e^{-z}z^{z-1/2}}=1. \tag{2.9}$$

Proof of Lemma 2.8. We can see that $\Psi_{\alpha,\beta}(\xi)$ is analytic in α,β for $-n<\operatorname{Re}(\alpha)<2$, $\operatorname{Re}(\beta)>0$, if ξ is fixed. By Lemma 2.9, $|\Psi_{\alpha,\beta}(\xi)|$ behaves like $|\xi|^{\operatorname{Re}(\alpha)-\operatorname{Re}(\beta)+n/2-1}$

if $|\xi|$ is sufficiently large. Also we note that $\Psi_{\alpha,\beta}(\xi)$ is continuous and does not vanish in α, β, ξ with $-n < \mathrm{Re}(\alpha) < 2$, $\mathrm{Re}(\beta) > 0$, $\xi \in \mathbb{R}^n$. Thus we have

$$A(1+|\xi|)^{\mathrm{Re}(\alpha)-\mathrm{Re}(\beta)+n/2-1} \le |\Psi_{\alpha,\beta}(\xi)| \le B(1+|\xi|)^{\mathrm{Re}(\alpha)-\mathrm{Re}(\beta)+n/2-1} \tag{2.10}$$

with some positive numbers A and B independent of ξ.

Using (2.10), since $\hat{g} \in \mathcal{S}(\mathbb{R})$ and $\widehat{\Theta}_\alpha$ is bounded and analytic for $\mathrm{Re}(\alpha) < 2$ by part (1) of Lemma 2.5, we can see $H(\alpha,\beta)$ is analytic for $-n < \mathrm{Re}(\alpha) < 2$, $\mathrm{Re}(\beta) > 0$. □

Now we are able to prove the following.

Proposition 2.10. *Let* $-n < \mathrm{Re}(\alpha) < 2$, $\mathrm{Re}(\beta) > 0$. *Then*

$$\sigma_\beta(I^\alpha f)(x)^2 = \int_{-\infty}^{\infty} |K_{\alpha,\beta} * \Theta_\alpha(u)|^2\, du = \int_{-\infty}^{\infty} |\Psi_{\alpha,\beta}(\xi)|^2 |\widehat{\Theta}_\alpha(\xi)|^2\, d\xi.$$

Proof. We recall that $G(\alpha,\beta)$ in (2.8) is analytic in α and β for $\mathrm{Re}(\alpha) > -n-2$, $\mathrm{Re}(\beta) > 0$ and that $H(\alpha,\beta)$ is analytic for $-n < \mathrm{Re}(\alpha) < 2$, $\mathrm{Re}(\beta) > 0$ (Lemma 2.8). Further, by Lemma 2.7 we have $G(\alpha,\beta) = H(\alpha,\beta)$ if $-n/2-1 < \mathrm{Re}(\alpha) < -n/2$ and $\mathrm{Re}(\beta) > 0$. Thus by analytic continuation we see that $G(\alpha,\beta) = H(\alpha,\beta)$ if $-n < \mathrm{Re}(\alpha) < 2$, $\mathrm{Re}(\beta) > 0$. So, we have $|G(\alpha,\beta)| = |H(\alpha,\beta)|$ if $-n < \mathrm{Re}(\alpha) < 2$, $\mathrm{Re}(\beta) > 0$. Thus, taking the supremum over g in the unit ball of $L^2(\mathbb{R})$ and recalling (2.7), we get the conclusion. □

On the other hand, for $D^\alpha(f)$ in (1.2) we have the following result.

Proposition 2.11. *Let* Θ_α *be as in Lemma* 2.5. *Suppose that* $0 < \mathrm{Re}(\alpha) < 2$. *Then*

$$D^\alpha(f)(x)^2 = \int_{-\infty}^{\infty} |\alpha - 2\pi i\xi|^{-2} |\widehat{\Theta}_\alpha(\xi)|^2\, d\xi.$$

Proof. We note that

$$\int_{S^{n-1}} (f(x-t\theta) - f(x))\, d\sigma(\theta) = \varphi(t;x,f) - \varphi(0;x,f) = \int_0^1 \theta(tr;x,f)\, \frac{dr}{r}$$

and

$$t^{-\alpha}(\varphi(t;x,f) - \varphi(0;x,f)) = \int_0^1 r^\alpha \theta_\alpha(tr;x,f)\, \frac{dr}{r}.$$

Thus

$$D^\alpha(f)(x) = \left(\int_0^\infty \left| \int_0^1 r^\alpha \theta_\alpha(tr;x,f)\, \frac{dr}{r} \right|^2 \frac{dt}{t} \right)^{1/2}.$$

By the change of variables $r = e^v$, $t = e^{-u}$, we have

$$D^\alpha(f)(x)^2 = \int_{-\infty}^{\infty} \left| \int_{-\infty}^0 e^{\alpha v} \Theta_\alpha(u-v)\, dv \right|^2 du = \int_{-\infty}^{\infty} |\hat{\psi}_\alpha(\xi)|^2 |\widehat{\Theta}_\alpha(\xi)|^2\, d\xi,$$

where $\psi_\alpha(u) = e^{\alpha u}\chi_{\{u \le 0\}}(u)$ and hence

$$\hat{\psi}_\alpha(\xi) = \int_{-\infty}^0 e^{\alpha u} e^{-2\pi i u\xi}\, du = \frac{1}{\alpha - 2\pi i\xi}.$$

Here we note that ψ_α and Θ_α are in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$ if $0 < \mathrm{Re}(\alpha) < 2$ (see Lemma 2.5 for Θ_α). This completes the proof. □

Proof of Theorem 1.1. Let $0 < \alpha < 2$. Then we note that

$$(\alpha/2)(1 + |\xi|) \leq |\alpha + 2\pi i\xi| \leq 2\pi(1 + |\xi|).$$

By this and (2.10) it follows that $|\alpha - 2\pi i\xi|^{-1}$ and $|\Psi_{\alpha,\beta}(\xi)|$ are pointwise equivalent as functions of ξ if $\beta = \alpha + n/2$. Using the pointwise equivalence and the formulas of Propositions 2.10 and 2.11, we have $\sigma_\beta(I^\alpha f)(x) \approx D^\alpha(f)(x)$ for $f \in \mathcal{S}_0(\mathbb{R}^n)$. Substituting $I_\alpha f$ for f and recalling (1.4), (2.3), we reach the conclusion. □

3. Applications

Let D^α, I_α and σ_β be as in (1.2), (1.4) and (1.3), respectively. Define

$$S_\alpha(f)(x) = D^\alpha(I_\alpha f)(x) \tag{3.1}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Then, some L^p estimates for S_α, $0 < \alpha < 2$, are shown in [7] with weights for $1/2 \leq \alpha < 2$, which are useful in characterizing the Sobolev spaces of order α (see [1] for relevant results). When $1 < \alpha < 2$, the result is proved by applying a theorem of [6] for the boundedness of Littlewood-Paley operators. When $0 < \alpha < 1$ it is shown by using Theorem 1.1 for $\alpha \in (0, 1)$, which is due to [4], and applying known properties of σ_β. The result for the case $\alpha = 1$ is due to [3]. Here, we focus on the case $1/2 \leq \alpha < 2$. Then, more precisely, we can find the following result in [7].

Theorem A. *Suppose that* $1/2 \leq \alpha < 2$, $w \in A_p$, $1 < p < \infty$. *Let* S_α *be as in* (3.1). *Let* $f \in \mathcal{S}(\mathbb{R}^n)$. *Then*

$$\|S_\alpha(f)\|_{p,w} \simeq \|f\|_{p,w}.$$

We recall the weight class A_p of Muckenhoupt. A weight w belongs to A_p, $1 < p < \infty$, if

$$\sup_B \left(|B|^{-1} \int_B w(x)\,dx\right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure of B (see [2] for the A_p class). The weighted L^p space is defined as L^p_w with the norm

$$\|f\|_{L^p_w} = \|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}.$$

Theorem A is due to [3] when $\alpha = 1$, as mentioned above. We can now give a different proof of this by applying Theorem 1.1 as follows. Since it is known that $\sigma_{n/2+1}$ is bounded on L^p_w for all $p \in (1, \infty)$ and $w \in A_p$ (see [7]), by Theorem 1.1 with $\alpha = 1$ we have

$$\|S_1(f)\|_{p,w} \leq C\|f\|_{p,w}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. The reverse inequality follows from this by duality as in [7].

Similarly, we can give another proof of Theorem A for $\alpha \in (1, 2)$, which was proved in [7] from a result of [6], by applying Theorem 1.1 and the boundedness of $\sigma_{n/2+\alpha}$ on L^p_w with $w \in A_p$, $1 < p < \infty$.

4. Proof of Lemma 2.9

By (2.9) to prove Lemma 2.9 it suffices to show

$$\exp\left(-\frac{a^3}{a^2+\xi^2}\right)\left(1+\frac{a^2}{\xi^2}\right)^{(a-1/2)/2} \le \frac{|e^{-z}z^{z-1/2}|}{e^{-\pi|\xi|/2}|\xi|^{a-1/2}} \le \left(1+\frac{a^2}{\xi^2}\right)^{(a-1/2)/2}, \tag{4.1}$$

where $z = a + i\xi$, $\xi \neq 0$.

To prove (4.1), we first note that

$$|z^{z-1/2}| = e^{(a-1/2)\log|z|}e^{-\xi \arg z} = (a^2+\xi^2)^{(a-1/2)/2}e^{-|\xi|\arg(a+i|\xi|)},$$

where $-\pi/2 < \arg z < \pi/2$. We write $\arg(a + i|\xi|) = \arctan(|\xi|/a)$. Define a function F on $[0,\infty)$ by $F(x) = \arctan(1/x)$, $x > 0$, $F(0) = \pi/2$. Then by the mean value theorem, we have

$$\pi/2 - \arctan(|\xi|/a) = F(0) - F(a/|\xi|) = \frac{a}{|\xi|}\frac{1}{\eta^2+1}$$

for some $\eta \in (0, a/|\xi|)$. Thus

$$e^{-a}e^{|\xi|(\pi/2-\arg(a+i|\xi|))} = e^{-a\eta^2/(\eta^2+1)}$$

and hence

$$\begin{aligned} |e^{-z}z^{z-1/2}| &= e^{-a}(a^2+\xi^2)^{(a-1/2)/2}e^{-\pi|\xi|/2}e^{|\xi|(\pi/2-\arg(a+i|\xi|))} \\ &= e^{-\pi|\xi|/2}|\xi|^{a-1/2}\left(1+\frac{a^2}{\xi^2}\right)^{(a-1/2)/2}e^{-a\eta^2/(\eta^2+1)}. \end{aligned} \tag{4.2}$$

Since

$$e^{-a^3/(a^2+\xi^2)} < e^{-a\eta^2/(\eta^2+1)} < 1,$$

from (4.2) we obtain (4.1). This completes the proof of Lemma 2.9.

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