Square functions related to integral of Marcinkiewicz and Sobolev spaces

SQUARE FUNCTIONS RELATED TO INTEGRAL OF MARCINKIEWICZ AND SOBOLEV SPACES

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Abstract. We prove a characterization of Sobolev spaces of order 2 by square functions related to the integral of Marcinkiewicz.

1. Introduction

Let ψ be a function in $L^1(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} \psi(x)\,dx = 0. \tag{1.1}$$

We consider the Littlewood-Paley function on $\mathbb{R}^n$ defined by

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \,\frac{dt}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(t^{-1}x)$, and a discrete parameter version of g_ψ:

$$\Delta_\psi(f)(x) = \left(\sum_{k=-\infty}^{\infty} |f * \psi_{2^k}(x)|^2\right)^{1/2}.$$

We recall the non-degeneracy conditions

$$\sup_{t>0} |\hat\psi(t\xi)| > 0 \quad \text{for all } \xi \neq 0; \tag{1.2}$$

$$\sup_{k\in\mathbb{Z}} |\hat\psi(2^k\xi)| > 0 \quad \text{for all } \xi \neq 0, \tag{1.3}$$

where $\mathbb{Z}$ denotes the set of integers and the Fourier transform $\hat\psi$ is defined by

$$\hat\psi(\xi) = \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i\langle x,\xi\rangle}\,dx, \quad \langle x,\xi\rangle = \sum_{k=1}^n x_k\xi_k.$$

Obviously, (1.3) implies (1.2). The weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ with a weight w is defined to be the class of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p} < \infty.$$

Then the following two theorems are known (see [11]).

Theorem A. *Suppose that*

(1) $B_\epsilon(\psi) < \infty$ *for some* $\epsilon > 0$, *where* $B_\epsilon(\psi) = \int_{|x|>1} |\psi(x)|\,|x|^\epsilon\,dx$;

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(2) $D_u(\psi) < \infty$ *for some* $u > 1$ *with* $D_u(\psi) = \left(\int_{|x|<1} |\psi(x)|^u\, dx\right)^{1/u}$;
(3) $H_\psi \in L^1(\mathbb{R}^n)$, *where* $H_\psi(x) = \sup_{|y|\ge|x|} |\psi(y)|$;
(4) *the non-degeneracy condition* (1.2) *holds.*

Then $\|f\|_{p,w} \simeq \|g_\psi(f)\|_{p,w}$, $f \in L^p_w$, *for all* $p \in (1,\infty)$ *and* $w \in A_p$ *(the Muckenhoupt class), where* $\|f\|_{p,w} \simeq \|g_\psi(f)\|_{p,w}$ *means that*

$$c_1\|f\|_{p,w} \le \|g_\psi(f)\|_{p,w} \le c_2\|f\|_{p,w}$$

with positive constants c_1, c_2 *independent of* f.

Theorem B. *We assume that*

(1) $B_\epsilon(\psi) < \infty$ *for some* $\epsilon > 0$;
(2) $|\hat{\psi}(\xi)| \le C|\xi|^{-\delta}$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$ *with some* $\delta > 0$;
(3) $H_\psi \in L^1(\mathbb{R}^n)$;
(4) *the non-degeneracy condition* (1.3) *holds.*

Then $\|f\|_{p,w} \simeq \|\Delta_\psi(f)\|_{p,w}$, $f \in L^p_w$, *for all* $p \in (1,\infty)$ *and* $w \in A_p$.

The inequality $\|g_\psi(f)\|_{p,w} \le c\|f\|_{p,w}$ in Theorem A was shown in [8] without the assumption (4).

The Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$, $\alpha > 0$, $1 < p < \infty$, consists of all the functions f which can be written as $f = J_\alpha(g) = K_\alpha * g$ for some $g \in L^p(\mathbb{R}^n)$ with the Bessel potential J_α, where

$$\hat{K}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}$$

(see [12, Chap. V]). The norm of f in $W^{\alpha,p}(\mathbb{R}^n)$ is defined as $\|f\|_{p,\alpha} = \|g\|_p$. Let $0 < \alpha < 2$. The operator

$$\mathcal{U}_\alpha(f)(x) = \left(\int_0^\infty \left| f(x) - \fint_{B(x,t)} f(y)\, dy \right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}$$

was studied in [1] and used to characterize the space $W^{\alpha,p}(\mathbb{R}^n)$. Here we write

$$\fint_{B(x,t)} f(y)\, dy = \frac{1}{|B(x,t)|}\int_{B(x,t)} f(y)\, dy,$$

where $|B(x,t)|$ is the Lebesgue measure of a ball $B(x,t)$ in $\mathbb{R}^n$ with center x and radius t.

We recall the weight class A_p of Muckenhoupt. A weight w belongs to A_p, $1 < p < \infty$, if

$$\sup_B \left(\fint_B w(x)\, dx\right)\left(\fint_B w(x)^{-1/(p-1)}\, dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ (see [4]).

Let $1 < p < \infty$, $\alpha > 0$ and $w \in A_p$. Then $J_\alpha(g) \in L^p_w$ if $g \in L^p_w$, since it is known that $|J_\alpha(g)| \le CM(g)$, where where M denotes the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{t>0} \fint_{B(x,t)} |f(y)|\, dy.$$

The weighted Sobolev space $W^{\alpha,p}_w(\mathbb{R}^n)$ is defined as the collection of all the functions $f \in L^p_w(\mathbb{R}^n)$ which can be expressed as $f = J_\alpha(g)$ for some $g \in L^p_w(\mathbb{R}^n)$; such g is uniquely determined and the norm is defined to be $\|f\|_{p,\alpha,w} = \|g\|_{p,w}$.

Theorems A, B can be applied to characterize the weighted Sobolev spaces $W_w^{\alpha,p}(\mathbb{R}^n)$ by square functions related to the Marcinkiewicz function including $\mathcal{U}_\alpha(f)$ and

$$\left(\sum_{k=-\infty}^{\infty}\left|f(x)-\fint_{B(x,2^k)} f(y)\,dy\right|^2 2^{-2k\alpha}\right)^{1/2}, \quad \alpha>0.$$

The Marcinkiewicz function was introduced by [7] (see [9] for some background materials).

We say $\Phi \in \mathcal{M}^\alpha(\mathbb{R}^n)$, $\alpha > 0$, if Φ is a compactly supported, bounded function on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n}\Phi(x)\,dx = 1$; if $\alpha \ge 1$, we further assume that

$$(1.4) \qquad \int_{\mathbb{R}^n}\Phi(x)x^\gamma\,dx = 0, \quad x^\gamma = x_1^{\gamma_1}\dots x_n^{\gamma_n}, \quad \text{for all } \gamma \text{ with } 1\le|\gamma|\le[\alpha],$$

where $\gamma = (\gamma_1,\dots,\gamma_n)$, $\gamma_j \in \mathbb{Z}$, $\gamma_j \ge 0$, is a multi-index and $|\gamma| = \gamma_1+\dots+\gamma_n$; also $[\alpha]$ denotes the largest integer not exceeding α. Let

$$(1.5) \qquad U_\alpha(f)(x) = \left(\int_0^\infty |f(x)-\Phi_t * f(x)|^2\frac{dt}{t^{1+2\alpha}}\right)^{1/2}, \quad \alpha>0,$$

$$(1.6) \qquad E_\alpha(f)(x) = \left(\sum_{k=-\infty}^{\infty}|f(x)-\Phi_{2^k}*f(x)|^2\,2^{-2k\alpha}\right)^{1/2}, \quad \alpha>0,$$

with $\Phi \in \mathcal{M}^\alpha(\mathbb{R}^n)$.

Then the following results are known (see [11]).

Theorem C. *Let $1<p<\infty$, $w\in A_p$ and $0<\alpha<n$. Let U_α be as in* (1.5). *Then $f \in W_w^{\alpha,p}(\mathbb{R}^n)$ if and only if $f\in L^p_w$ and $U_\alpha(f)\in L^p_w$; furthermore,*

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|U_\alpha(f)\|_{p,w}.$$

Theorem D. *Suppose that $1<p<\infty$, $w\in A_p$ and $0<\alpha<n$. Let E_α be as in* (1.6). *Then $f\in W_w^{\alpha,p}(\mathbb{R}^n)$ if and only if $f\in L^p_w$ and $E_\alpha(f)\in L^p_w$; also,*

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|E_\alpha(f)\|_{p,w}.$$

See [6, 10] for relevant results.

In this note we consider another characterization of $W_w^{2,p}(\mathbb{R}^n)$ by certain square functions relative to the integral of Marcinkiewicz when $n\ge 3$, which extends to the cases $n=1,2$.

Let $\Phi\in\mathcal{M}^1(\mathbb{R}^n)$. We assume

$$(1.7) \qquad \int_{\mathbb{R}^n}\Phi(x)x_j^2\,dx = \frac{1}{n}\int_{\mathbb{R}^n}\Phi(x)|x|^2\,dx = b_0 \quad \text{for all } j, 1\le j\le n.$$

When $n\ge 2$, we also assume

$$(1.8) \qquad \int_{\mathbb{R}^n}\Phi(x)x_jx_k\,dx = 0 \quad \text{for all } j,k,\ 1\le j,k\le n \text{ with } j\ne k.$$

Let I_α be the Riesz potential operator defined by

$$(1.9) \qquad \widehat{I_\alpha(f)}(\xi) = (2\pi|\xi|)^{-\alpha}\hat{f}(\xi), \quad 0<\alpha<n.$$

Let $L_\alpha(x) = \tau(\alpha)|x|^{\alpha-n}$, where

$$\tau(\alpha) = \frac{\Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right)}{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}.$$

Then $\widehat{L}_\alpha(\xi) = (2\pi|\xi|)^{-\alpha}$, $0 < \alpha < n$.

Let $n \geq 3$. Define

$$\psi(x) = \Phi * L_2(x) - L_2(x) + c_0\Phi(x) \tag{1.10}$$

with $c_0 = b_0/2$ and $\Phi \in \mathcal{M}^1(\mathbb{R}^n)$ satisfying (1.7) and (1.8); when $n = 1$ and $n = 2$, we have analogues of (1.10) in (5.5) and (4.4) below, respectively. Applying Theorems A and B, we have the following results.

Theorem 1.1. *Suppose that $n \geq 3$. Let $w \in A_p$, $p \in (1, \infty)$. Let ψ be as in* (1.10) *with $\Phi \in \mathcal{M}^1(\mathbb{R}^n)$ satisfying* (1.7) *and* (1.8). *Suppose that the non-degeneracy condition* (1.2) *holds. Then*

$$\|f\|_{p,w} \simeq \|g_\psi(f)\|_{p,w}, \quad f \in L^p_w.$$

Theorem 1.2. *Let $n \geq 3$. Let Φ be a function in $\mathcal{M}^1(\mathbb{R}^n)$ with* (1.7), (1.8) *and let ψ be as in* (1.10). *We assume that*

$$|\hat{\Phi}(\xi)| \leq C|\xi|^{-\delta} \quad \textit{for all } \xi \in \mathbb{R}^n \setminus \{0\} \textit{ with some } \delta > 0 \tag{1.11}$$

and that the non-degeneracy condition (1.3) *holds. Then we have*

$$\|f\|_{p,w} \simeq \|\Delta_\psi(f)\|_{p,w}, \quad f \in L^p_w$$

for all $p \in (1, \infty)$ and $w \in A_p$.

Theorems 1.1 and 1.2 will be used to prove Theorems 1.4 and 1.5 below for $n \geq 3$, respectively.

Proof of Theorem 1.1. Suppose that $\operatorname{supp}(\Phi) \subset \{|x| \leq M\}$. Then we have $|\psi(x)| \leq C|x|^{2-n}$ if $|x| \leq 2M$. Let $|x| \geq 2M$. Then, applying Taylor's formula, by (1.7), (1.8) and (1.4) with $|\gamma| = 1$ we see that

$$\begin{aligned} L_2 * \Phi(x) - L_2(x) &= \tau(2)\int_{\mathbb{R}^n} (|x-y|^{2-n} - |x|^{2-n})\Phi(y)\,dy \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \sum_{j=1}^n y_j^2 \partial_j^2 L_2(x)\Phi(y)\,dy + O(|x|^{-n-1}) \\ &= \frac{1}{2} b_0 \sum_{j=1}^n \partial_j^2 L_2(x) + O(|x|^{-n-1}) \\ &= O(|x|^{-n-1}), \end{aligned}$$

as $|x| \to \infty$, where the last equality follows from $\Delta L_2(x) = \sum_{j=1}^n \partial_j^2 L_2(x) = 0$, $\partial_j = \partial/\partial x_j$.

We see that

$$\hat{\psi}(\xi) = (2\pi|\xi|)^{-2}\hat{\Phi}(\xi) - (2\pi|\xi|)^{-2} + c_0\hat{\Phi}(\xi) = (2\pi|\xi|)^{-2}(\hat{\Phi}(\xi) - 1) + c_0\hat{\Phi}(\xi).$$

Also, by (1.7), (1.8) and (1.4) with $|\gamma| = 1$, we have

$$\begin{aligned}\hat{\Phi}(\xi) &= \int_{\mathbb{R}^n} \Phi(x)e^{-2\pi i\langle x,\xi\rangle}\,dx \\ &= 1 + \int_{\mathbb{R}^n} \Phi(x)\frac{1}{2}(-2\pi i\langle x,\xi\rangle)^2\,dx + O(|\xi|^3) \\ &= 1 - 2\pi^2\int_{\mathbb{R}^n} \Phi(x)(\sum_{j=1}^n x_j^2\xi_j^2)\,dx + O(|\xi|^3) \\ &= 1 - 2\pi^2 b_0|\xi|^2 + O(|\xi|^3),\end{aligned}$$

as $|\xi| \to 0$. Thus, since $c_0 = b_0/2$, we have $|\hat{\psi}(\xi)| \le C|\xi|$ and hence (1.1). Altogether, thus we can apply Theorem A to get the conclusion of Theorem 1.1. $\square$

Similarly, Theorem 1.2 follows from Theorem B.

Define $\mathcal{L} = -\Delta = -\sum_{j=1}^n \partial_j^2$, $\partial_j = \partial/\partial x_j$, on $\mathbb{R}^n$, $n \ge 1$. Then, if $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\widehat{\mathcal{L}(f)}(\xi) = (2\pi|\xi|)^2\hat{f}(\xi),$$

where we have denoted by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$. We note the following.

Lemma 1.3. *Let $n \ge 1$. Define H_0 on $\mathcal{S}(\mathbb{R}^n)$ by $H_0(f) = \mathcal{L}(J_2(f))$. Then H_0 extends to a bounded operator on L^p_w and also we have $H_0(f) = \mathcal{L}(J_2(f))$ for $f \in L^p_w$, where $\mathcal{L} = -\Delta = -\sum_{j=1}^n \partial_j^2$ is defined by the weak derivative:*

$$\int_{\mathbb{R}^n} H_0(f)(x)\eta(x)\,dx = \int_{\mathbb{R}^n} J_2(f)(x)\mathcal{L}(\eta)(x)\,dx = -\int_{\mathbb{R}^n} J_2(f)(x)\sum_{j=1}^n \partial_j^2\eta(x)\,dx$$

for all $\eta \in \mathcal{S}(\mathbb{R}^n)$.

We shall give a proof of Lemma 1.3 in Section 2.

Let $\Phi \in \mathcal{M}^1(\mathbb{R}^n)$. Let

$$S(f)(x) = \left(\int_0^\infty |f * \Phi_t(x) - f(x) + c_0t^2\mathcal{L}(f) * \Phi_t(x)|^2\,\frac{dt}{t^5}\right)^{1/2}, \tag{1.12}$$

when $f, \mathcal{L}(f) \in L^p_w$, where c_0 is as in (1.10). For $g \in L^p_w$ let $H_0(g)$ be as in Lemma 1.3 and define

$$S_2(g)(x) = \left(\int_0^\infty |J_2(g) * \Phi_t(x) - J_2(g)(x) + c_0t^2H_0(g) * \Phi_t(x)|^2\,\frac{dt}{t^5}\right)^{1/2}. \tag{1.13}$$

Then $S(J_2(g)) = S_2(g)$ for $g \in L^p_w$ by Lemma 1.3. Let

$$S(f,g)(x) = \left(\int_0^\infty |f * \Phi_t(x) - f(x) + c_0t^2g * \Phi_t(x)|^2\,\frac{dt}{t^5}\right)^{1/2} \tag{1.14}$$

for $f, g \in L^p_w$. Then, if $f, \mathcal{L}(f) \in L^p_w$, we have $S(f, \mathcal{L}(f)) = S(f)$.

The square function $S(f,g)$ is able to characterize the space $W^{2,p}_w$ as follows.

Theorem 1.4. *Let $n \ge 1$. Suppose that $f \in L^p_w$, $1 < p < \infty$, $w \in A_p$. Let $S(f)$, $S(f,g)$ be as in* (1.12), (1.14)*, respectively, with $\Phi \in \mathcal{M}^1(\mathbb{R}^n)$ satisfying* (1.7), (1.8) *and* (1.2)*, where Φ and ψ are related as in* (1.10), (4.4) *or* (5.5) *according as $n \ge 3$, $n = 2$ or $n = 1$. Then*

(1) *if $f \in W^{2,p}_w$, then $\mathcal{L}(f) \in L^p_w$ and $S(f) \in L^p_w$;*

(2) *if* $S(f,g) \in L^p_w$ *for some* $g \in L^p_w$, *then* $f \in W^{2,p}_w$ *and* $g = \mathcal{L}(f)$.
Also, if $f \in W^{2,p}_w$,

$$\|S(f)\|_{p,w} \simeq \|\mathcal{L}(f)\|_{p,w}, \quad \|S(f)\|_{p,w} + \|f\|_{p,w} \simeq \|f\|_{p,2,w}.$$

We can also consider discrete parameter version of Theorem 1.4. Let $\Phi \in \mathcal{M}^1(\mathbb{R}^n)$ and

$$(1.15) \quad V(f)(x) = \left(\sum_{k=-\infty}^{\infty} |f * \Phi_{2^k}(x) - f(x) + c_0 2^{2k}\mathcal{L}(f) * \Phi_{2^k}(x)|^2 2^{-4k}\right)^{1/2},$$

if $f, \mathcal{L}(f) \in L^p_w$. Let

(1.16)

$$V_2(g)(x) = \left(\sum_{k=-\infty}^{\infty} |J_2(g) * \Phi_{2^k}(x) - J_2(g)(x) + c_0 2^{2k} H_0(g) * \Phi_{2^k}(x)|^2 2^{-4k}\right)^{1/2}$$

for $g \in L^p_w$. If $g \in L^p_w$, we have $V(J_2(g)) = V_2(g)$ by Lemma 1.3. For $f, g \in L^p_w$, let

$$(1.17) \quad V(f,g)(x) = \left(\sum_{k=-\infty}^{\infty} |f * \Phi_{2^k}(x) - f(x) + c_0 2^{2k} g * \Phi_{2^k}(x)|^2 2^{-4k}\right)^{1/2}.$$

We have $V(f, \mathcal{L}(f)) = V(f)$ if $f, \mathcal{L}(f) \in L^p_w$.

We have a discrete parameter analogue of Theorem 1.4.

Theorem 1.5. *Suppose that* $n \geq 1$ *and* $f \in L^p_w$, $1 < p < \infty$, $w \in A_p$. *Let* Φ *be a function in* $\mathcal{M}^1(\mathbb{R}^n)$ *satisfying* (1.7), (1.8), (1.11) *and* (1.3), *where* Φ *and* ψ *are related as in Theorem* 1.4. *Let* $V(f)$ *and* $V(f,g)$ *be as in* (1.15) *and* (1.17), *respectively. Then*

(1) $\mathcal{L}(f) \in L^p_w$ *and* $V(f) \in L^p_w$ *if* $f \in W^{2,p}_w$;

(2) *if* $V(f,g) \in L^p_w$ *for some* $g \in L^p_w$, *it follows that* $f \in W^{2,p}_w$ *and* $g = \mathcal{L}(f)$.
Further, if $f \in W^{2,p}_w$,

$$\|V(f)\|_{p,w} \simeq \|\mathcal{L}(f)\|_{p,w}, \quad \|V(f)\|_{p,w} + \|f\|_{p,w} \simeq \|f\|_{p,2,w}.$$

See [2] for characterization of the Sobolev spaces by square functions related to the Lusin area integral and the Littlewood-Paley g^*_λ function.

Let Φ be a function in $\mathcal{M}^1(\mathbb{R}^n)$ satisfying (1.7) and (1.8), then we have already seen in the proof of Theorem 1.1 that the function ψ defined by (1.10), $n \geq 3$, satisfies the conditions (1.1) and (1), (2), (3) of Theorem A. This is also the case for functions ψ in (4.4) and in (5.5) below, on $\mathbb{R}^2$ and on $\mathbb{R}$, respectively, as can be shown similarly.

Let us further assume that Φ is a radial function. Then, we have the decay estimate (1.11) by the formula in [13, p.155, Theorem 3.3] for $n \geq 2$. Also, if Φ is a radial function, it follows that ψ defined by (1.10) satisfies the non-degeneracy condition (1.3) and hence (1.2). This is also the case for functions ψ in (4.4) and (5.5).

We can see (1.3) when Φ is a radial function as follows. First, we note that there exists an entire function $G(z) = \sum_{k=1}^{\infty} a_k z^k$ such that $\hat{\psi}(\xi) = G(|\xi|)$. We can see that ψ is not identically 0. This holds since ψ is unbounded when $n \geq 2$; the result for $n = 1$ is also seen by an inspection (see Section 5). Therefore we have (1.3) since $z = 0$ cannot be an accumulation point of zeros of $G(z)$.

If $\Phi = |B(0,1)|^{-1}\chi_{B(0,1)}$, then $\Phi \in \mathcal{M}^1(\mathbb{R}^n)$ and Φ satisfies (1.7) with $b_0 = 2c_0 = 1/(n+2)$, (1.8), (1.11) and (1.3) with ψ as in (1.10), (4.4) and (5.5), for all $n \geq 1$. This follows from remarks above and easy observations. In this case we can rewrite $S(f)$, $S(f,g)$ and $V(f)$, $V(f,g)$ as follows.

$$S(f)(x)^2 = \int_0^\infty \left| \fint_{B(x,t)} \left(f(y) - f(x) - \frac{1}{2n}(\Delta f)_{B(x,t)}|y-x|^2 \right) dy \right|^2 \frac{dt}{t^5};$$

$$S(f,g)(x)^2 = \int_0^\infty \left| \fint_{B(x,t)} \left(f(y) - f(x) + \frac{1}{2n}g_{B(x,t)}|y-x|^2 \right) dy \right|^2 \frac{dt}{t^5};$$

$$V(f)(x)^2 = \sum_{k=-\infty}^{\infty} \left| \fint_{B(x,2^k)} \left(f(y) - f(x) - \frac{1}{2n}(\Delta f)_{B(x,2^k)}|y-x|^2 \right) dy \right|^2 2^{-4k};$$

$$V(f,g)(x)^2 = \sum_{k=-\infty}^{\infty} \left| \fint_{B(x,2^k)} \left(f(y) - f(x) + \frac{1}{2n}g_{B(x,2^k)}|y-x|^2 \right) dy \right|^2 2^{-4k},$$

where $f_B = \fint_B f$. The square functions $S(f)$, $S(f,g)$ are considered in [1] and unweighted results concerning them contained in Theorem 1.4 are due to [1].

In Section 2, we shall prove Lemma 1.3 and Theorem 1.4 for $n \geq 3$ by applying Theorem 1.1. Theorem 1.5 can be proved in the same way as Theorem 1.4, by using Theorem 1.2 if $n \geq 3$. We shall give an outline of the proof of Theorem 1.5 for $n \geq 3$ in Section 3.

To prove Theorems 1.4 and 1.5 for $n = 1, 2$, we need analogues of Theorems 1.1 and 1.2. The cases $n = 1, 2$ should be treated separately, since the Riesz potential is not available as in the case of $\mathbb{R}^n$ above for $n \geq 3$. In Section 4, in the two dimensional case, Theorems 1.4 and 1.5 will be proved, where analogues of Theorems 1.1 and 1.2 will be shown for $n = 2$. Finally, in Section 5, we shall prove Theorems 1.4 and 1.5 for $n = 1$. Also, analogues of Theorems 1.1 and 1.2 for $n = 1$ will be given.

2. Proof of Theorem 1.4 for $n \geq 3$

We need the following.

Lemma 2.1. *Let S and S_2 be as in* (1.12) *and* (1.13)*, respectively, on $\mathbb{R}^n$, $n \geq 1$, with Φ as in Theorem* 1.4*. Let $g \in L^p_w$, $w \in A_p$, $1 < p < \infty$. Then*

$$\|S(J_2(g))\|_{p,w} + \|J_2(g)\|_{p,w} = \|S_2(g)\|_{p,w} + \|J_2(g)\|_{p,w} \simeq \|g\|_{p,w}. \tag{2.1}$$

We give a proof of Lemma 2.1 for $n \geq 3$ in this section. The results for $n = 2$ and $n = 1$ can be shown similarly with the arguments in Sections 4 and 5, respectively.

The following relations concerning Riesz and Bessel potentials are useful.

Lemma 2.2. *Let $\alpha > 0$. Suppose that $1 < p < \infty$ and w is a weight in A_p on $\mathbb{R}^n$, $n \geq 1$.*

(1) *We can find a Fourier multiplier ℓ for L^p_w such that*

$$(2\pi|\xi|)^\alpha = \ell(\xi)(1 + 4\pi^2|\xi|^2)^{\alpha/2}.$$

(2) *We have*

$$(1 + 4\pi^2|\xi|^2)^{\alpha/2} = m(\xi) + m(\xi)(2\pi|\xi|)^\alpha$$

with some Fourier multiplier m for L^p_w.

Here we give a proof of Lemma 1.3.

Proof of Lemma 1.3 . By part (1) of Lemma 2.2, we see that H_0 initially defined on $\mathcal{S}(\mathbb{R}^n)$ extends to a bounded operator on L^p_w and integration by parts implies

$$\int_{\mathbb{R}^n} H_0(f)(x)\eta(x)\,dx = -\int_{\mathbb{R}^n} J_2(f)(x)\sum_{j=1}^n \partial_j^2\eta(x)\,dx$$

for all $\eta \in \mathcal{S}(\mathbb{R}^n)$ if $f \in \mathcal{S}(\mathbb{R}^n)$. Since both sides of the equality above are continuous in $f \in L^p_w$ for each fixed η and $\mathcal{S}(\mathbb{R}^n)$ is dense in L^p_w, we get the conclusion. □

Proof of Lemma 2.1 *for* $n \geq 3$. We first prove (2.1) for $g \in \mathcal{S}(\mathbb{R}^n)$. We can write

$$S_2(g) = g_\psi(H_0(g)).$$

Thus Theorem 1.1 implies

$$\|S_2(g)\|_{p,w} = \|g_\psi(H_0(g))\|_{p,w} \simeq \|H_0(g)\|_{p,w} \leq C\|g\|_{p,w}. \tag{2.2}$$

Also, by part (2) of Lemma 2.2 and Theorem 1.1

$$\begin{aligned}\|g\|_{p,w} &= \|J_{-2}J_2(g)\|_{p,w} \leq C\|J_2(g)\|_{p,w} + C\|\mathcal{L}J_2(g)\|_{p,w} \\ &\leq C\|J_2(g)\|_{p,w} + C\|S_2(g)\|_{p,w}.\end{aligned} \tag{2.3}$$

From (2.2) and (2.3), (2.1) follows for $g \in \mathcal{S}(\mathbb{R}^n)$.

Let

$$S_2^N(g)(x) = \left(\int_{N^{-1}}^N |J_2(g)*\Phi_t(x) - J_2(g)(x) + c_0t^2H_0(g)*\Phi_t(x)|^2\,\frac{dt}{t^5}\right)^{1/2}.$$

Then $\|S_2^N(g)\|_{p,w} \leq C_N\|g\|_{p,w}$ for $g \in L^p_w$. Using this and (2.1) for $g \in \mathcal{S}(\mathbb{R}^n)$, we have $\|S_2^N(g)\|_{p,w} \leq C\|g\|_{p,w}$ for $g \in L^p_w$ with a constant C independent of N, since $\mathcal{S}(\mathbb{R}^n)$ is dense in L^p_w. Thus, letting $N \to \infty$, we have $\|S_2(g)\|_{p,w} \leq C\|g\|_{p,w}$ for $g \in L^p_w$. We can take a sequence $\{g_k\}$ in $\mathcal{S}(\mathbb{R}^n)$ such that $g_k \to g$ in L^p_w and $J_2(g_k) \to J_2(g)$ in L^p_w as $k \to \infty$. Then we note that $\|S_2(g_k)\|_{p,w} \to \|S_2(g)\|_{p,w}$. Thus, letting $k \to \infty$ in the relation

$$\|S_2(g_k)\|_{p,w} + \|J_2(g_k)\|_{p,w} \simeq \|g_k\|_{p,w},$$

which has been already shown, we get the conclusion. □

The next result will be useful in what follows (see [11] for a proof).

Lemma 2.3. *Suppose that* f *is in* L^p_w *on* $\mathbb{R}^n$, $n \geq 1$, *with* $w \in A_p$, $1 < p < \infty$. *Let* $g \in \mathcal{S}(\mathbb{R}^n)$ *and* $\alpha > 0$. *Then we have*

(1) $K_\alpha * (f*g)(x) = (K_\alpha * f)*g(x) = (K_\alpha * g)*f(x)$ *for every* $x \in \mathbb{R}^n$;
(2) $\int_{\mathbb{R}^n}(K_\alpha * f)(y)g(y)\,dy = \int_{\mathbb{R}^n}(K_\alpha * g)(y)f(y)\,dy$.

Proof of Theorem 1.4 *for* $n \geq 3$. If $f \in W^{2,p}_w$, $f = J_2(g)$ for some $g \in L^p_w$. Thus by Lemma 1.3 and Lemma 2.1 we have part (1).

Suppose $f, g, S(f,g) \in L^p_w$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\int \varphi = 1$ and put $f^\epsilon = f*\varphi_\epsilon$, $g^\epsilon = g*\varphi_\epsilon$, $h^\epsilon = f*J_{-2}(\varphi_\epsilon)$. We note that $f^\epsilon = J_2(h^\epsilon)$ by Lemma 2.3, $f^\epsilon, g^\epsilon, h^\epsilon \in L^p_w$ and $\mathcal{L}(f^\epsilon) = H_0(h^\epsilon)$ by Lemma 1.3. Also, $g^\epsilon \to g$, $f^\epsilon \to f$ in L^p_w.

By Minkowski's inequality we have

$$S(f^\epsilon, g^\epsilon)(x) \leq CM(S(f,g))(x). \tag{2.4}$$

Thus, since

$$\left(\int_0^\infty |c_0 H_0(h^\epsilon) * \Phi_t(x) - c_0 g^\epsilon * \Phi_t(x)|^2 \, \frac{dt}{t}\right)^{1/2} \le S_2(h^\epsilon)(x) + S(f^\epsilon, g^\epsilon)(x),$$

we see that the quantity on the left hand side belongs to L^p_w by (2.4) and Lemma 2.1. Thus

$$0 = \lim_{t\to 0} |H_0(h^\epsilon) * \Phi_t(x) - g^\epsilon * \Phi_t(x)| = |H_0(h^\epsilon)(x) - g^\epsilon(x)|,$$

which implies

$$H_0(h^\epsilon)(x) = g^\epsilon(x), \tag{2.5}$$

$$S_2(h^\epsilon)(x) = S(f^\epsilon, g^\epsilon)(x),$$

for almost every $x \in \mathbb{R}^n$, and hence

$$\|S_2(h^\epsilon)\|_{p,w} \le C$$

with a constant C independent of $\epsilon > 0$ by (2.4). Thus we have $\|h^\epsilon\|_{p,w} \simeq \|f^\epsilon\|_{p,w} + \|S_2(h^\epsilon)\|_{p,w} \le C$ by Lemma 2.1.

So, we have a sequence $\{h^{\epsilon_k}\}$ and $h \in L^p_w$ such that $h^{\epsilon_k} \to h$ weakly in L^p_w. For $\eta \in \mathcal{S}(\mathbb{R}^n)$, by (2.5), Lemma 1.3 and Lemma 2.3 we have

$$\begin{aligned}\int_{\mathbb{R}^n} H_0(h)\eta \, dx &= \int_{\mathbb{R}^n} J_2(h)\mathcal{L}(\eta)\, dx = \int_{\mathbb{R}^n} hJ_2(\mathcal{L}(\eta))\, dx \\ &= \lim_k \int_{\mathbb{R}^n} h^{\epsilon_k} J_2(\mathcal{L}(\eta))\, dx = \lim_k \int_{\mathbb{R}^n} J_2(h^{\epsilon_k})\mathcal{L}(\eta)\, dx \\ &= \lim_k \int_{\mathbb{R}^n} H_0(h^{\epsilon_k})\eta \, dx = \lim_k \int_{\mathbb{R}^n} g^{\epsilon_k}\eta\, dx = \int_{\mathbb{R}^n} g\eta\, dx.\end{aligned}$$

Thus $H_0(h) = g$. Also,

$$\int_{\mathbb{R}^n} H_0(h)\eta\, dx = \lim_k \int_{\mathbb{R}^n} J_2(h^{\epsilon_k})\mathcal{L}(\eta)\, dx = \lim_k \int_{\mathbb{R}^n} f^{\epsilon_k}\mathcal{L}(\eta)\, dx = \int_{\mathbb{R}^n} f\mathcal{L}(\eta)\, dx.$$

So we have $H_0(h) = g = \mathcal{L}(f)$. Similarly, we see that $f = J_2(h)$. This proves part (2).

By (2.2)

$$\|S_2(g)\|_{p,w} \simeq \|H_0(g)\|_{p,w} \tag{2.6}$$

for $g \in \mathcal{S}(\mathbb{R}^n)$. Since S_2 and H_0 are continuous on L^p_w and $\mathcal{S}(\mathbb{R}^n)$ is dense in L^p_w, we have (2.6) for all $g \in L^p_w$. If $f \in W^{2,p}_w$ and $f = J_2(h)$ with $h \in L^p_w$, $H_0(h) = \mathcal{L}(f)$ by Lemma 1.3 and $\|S_2(h)\|_{p,w} = \|S(f)\|_{p,w} \simeq \|\mathcal{L}(f)\|_{p,w}$ from (2.6). Also, by Lemma 2.1, $\|S(f)\|_{p,w} + \|f\|_{p,w} \simeq \|h\|_{p,w} = \|f\|_{p,2,w}$. This completes the proof of Theorem 1.4. □

3. Proof of Theorem 1.5 for $n \ge 3$

We can prove Theorem 1.5 similarly to the proof of Theorem 1.4. So, only the outline of the proof is given.

Lemma 3.1. *Let V and V_2 be as in* (1.15) *and* (1.16) *on $\mathbb{R}^n$, $n \ge 1$, respectively, with Φ as in Theorem* 1.5. *Suppose that $g \in L^p_w$, $w \in A_p$, $1 < p < \infty$. Then*

$$\|V(J_2(g))\|_{p,w} + \|J_2(g)\|_{p,w} = \|V_2(g)\|_{p,w} + \|J_2(g)\|_{p,w} \simeq \|g\|_{p,w}.$$

To prove Lemma 3.1 for $n \geq 3$ we note that

$$V_2(g) = \Delta_\psi(H_0(g))$$

for $g \in \mathcal{S}(\mathbb{R}^n)$ and apply Theorem 1.2 and Lemma 2.2.

Lemma 1.3 and Lemma 3.1 imply part (1) of Theorem 1.5. To prove part (2) of Theorem 1.5, let $f, g, V(f,g) \in L^p_w$ and f^ϵ, g^ϵ, h^ϵ be as in the proof of Theorem 1.4. Then

$$V(f^\epsilon, g^\epsilon)(x) \leq CM(V(f,g))(x)$$

by Minkowski's inequality. Using this and

$$\left(\sum_{k=-\infty}^{\infty} |c_0 H_0(h^\epsilon) * \Phi_{2^k}(x) - c_0 g^\epsilon * \Phi_{2^k}(x)|^2\right)^{1/2} \leq V_2(h^\epsilon)(x) + V(f^\epsilon, g^\epsilon)(x),$$

we can proceed as in the proof of Theorem 1.4 to get the assertion of part (2).

4. Two dimensional case

We consider $L_\alpha(x) = \tau(\alpha)|x|^{\alpha-2}$ on $\mathbb{R}^2$. Then we have the following (see [3, p. 151]).

Lemma 4.1. *For $\varphi \in \mathcal{S}(\mathbb{R}^2)$ we have*

$$\langle -\frac{1}{2\pi}\log|x|, \hat{\varphi}\rangle = \int_{\mathbb{R}^2}(-\frac{1}{2\pi}\log|x|)\hat{\varphi}(x)\,dx = \lim_{\substack{\alpha\to 2\\ \alpha<2}}\langle L_\alpha - \tau(\alpha), \hat{\varphi}\rangle$$

$$= \int_{|\xi|<1}(2\pi|\xi|)^{-2}(\varphi(\xi)-\varphi(0))\,d\xi + \int_{|\xi|\geq 1}(2\pi|\xi|)^{-2}\varphi(\xi)\,d\xi + \frac{1}{2\pi}\varphi(0)(-\Gamma'(1)+\log\pi).$$

It is known that $\Gamma'(1) = -\gamma$, where γ denotes Euler's constant.

Proof of Lemma 4.1. Let $\alpha \in (0,2)$. Then

$$\int_{|\xi|<1}(2\pi|\xi|)^{-\alpha}\,d\xi - \tau(\alpha) = \frac{(2\pi)^{1-\alpha}}{2-\alpha} - \frac{\Gamma\left(1-\frac{1}{2}\alpha\right)}{\Gamma\left(\frac{1}{2}\alpha\right)2^\alpha\pi} = (2\pi)^{1-\alpha}\frac{G(2)-G(\alpha)}{2-\alpha},$$

where

$$G(\alpha) = \frac{\Gamma\left(2-\frac{1}{2}\alpha\right)\pi^{\alpha-2}}{\Gamma\left(\frac{1}{2}\alpha\right)}.$$

We note that

$$G'(\alpha) = \frac{-\frac{1}{2}\Gamma'\left(2-\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}\alpha\right) - \frac{1}{2}\Gamma\left(2-\frac{1}{2}\alpha\right)\Gamma'\left(\frac{1}{2}\alpha\right)}{\Gamma\left(\frac{1}{2}\alpha\right)^2}\pi^{\alpha-2} - \frac{\Gamma\left(2-\frac{1}{2}\alpha\right)}{\Gamma\left(\frac{1}{2}\alpha\right)}\pi^{\alpha-2}\log\pi.$$

Thus

$$\int_{|\xi|<1}(2\pi|\xi|)^{-\alpha}\,d\xi - \tau(\alpha) \to \frac{-\Gamma'(1)+\log\pi}{2\pi} \quad \text{as } \alpha\to 2 \text{ with } \alpha<2. \tag{4.1}$$

On the other hand,

$$L_\alpha(x) - \tau(\alpha) = \frac{2\Gamma\left(2-\frac{1}{2}\alpha\right)}{\Gamma\left(\frac{1}{2}\alpha\right)2^\alpha\pi}\frac{|x|^{\alpha-2}-1}{2-\alpha} \to -\frac{1}{2\pi}\log|x| \quad \text{for } x \in \mathbb{R}^2\setminus\{0\} \tag{4.2}$$

as $\alpha \to 2$ with $\alpha < 2$. Also, if $\alpha \in (3/2, 2)$,

$$|L_\alpha(x) - \tau(\alpha)| \le C|x|^{-1}\chi_{B(0,2)}(x) + C|\log|x||\chi_{\mathbb{R}^2\setminus B(0,2)}(x) \tag{4.3}$$

with a constant C independent of α. By (4.1), (4.2), (4.3) and the Lebesgue convergence theorem we have

$$\begin{aligned}
&\langle -\frac{1}{2\pi}\log|x|, \hat{\varphi}\rangle = \lim_{\substack{\alpha\to 2\\ \alpha<2}} \langle L_\alpha - \tau(\alpha), \hat{\varphi}\rangle = \lim_{\substack{\alpha\to 2\\ \alpha<2}} \left(\int_{\mathbb{R}^2} (2\pi|\xi|)^{-\alpha}\varphi(\xi)\, d\xi - \tau(\alpha)\varphi(0)\right) \\
&= \lim_{\substack{\alpha\to 2\\ \alpha<2}} \left[\int_{|\xi|<1} (2\pi|\xi|)^{-\alpha}(\varphi(\xi) - \varphi(0))\, d\xi + \int_{|\xi|\ge 1} (2\pi|\xi|)^{-\alpha}\varphi(\xi)\, d\xi \right.\\
&\quad \left. + \varphi(0)\left(\int_{|\xi|<1} (2\pi|\xi|)^{-\alpha}\, d\xi - \tau(\alpha)\right)\right] \\
&= \int_{|\xi|<1} (2\pi|\xi|)^{-2}(\varphi(\xi) - \varphi(0))\, d\xi + \int_{|\xi|\ge 1} (2\pi|\xi|)^{-2}\varphi(\xi)\, d\xi \\
&\quad + \frac{1}{2\pi}\varphi(0)\left(-\Gamma'(1) + \log\pi\right).
\end{aligned}$$

□

Lemma 4.2. *Let* $L_2(x) = -\frac{1}{2\pi}\log|x|$ *on* $\mathbb{R}^2$. *Let* $\Phi \in \mathcal{M}^1(\mathbb{R}^2)$. *Suppose that* Φ *satisfies* (1.7), (1.8) *and* $\operatorname{supp}\Phi \subset \{|x| \le M\}$. *Let* $\eta(x) = L_2 * \Phi(x) - L_2(x)$. *Then* $|\eta(x)| \le C(1 + |\log|x||)$ *if* $|x| \le 2M$ *and* $|\eta(x)| \le C|x|^{-3}$ *if* $|x| \ge 2M$. *Also,* $\hat{\eta}(\xi) = (2\pi|\xi|)^{-2}(\hat{\Phi}(\xi) - 1)$.

Proof. The estimates $|\eta(x)| \le C(1+|\log|x||)$ for $|x| \le 2M$ and $|\eta(x)| \le C|x|^{-3}$ for $|x| \ge 2M$ can be shown as in the proof of Theorem 1.1, since $\Delta L_2 = 0$ on $\mathbb{R}^2 \setminus \{0\}$.

Let $\Psi \in C_0^\infty(\mathbb{R}^2)$ with $\Psi(0) = 1$. Let $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and $\varphi_{(\epsilon)}(\xi) = \varphi(\xi) - \varphi(0)\Psi(\xi/\epsilon)$. Then, since $\varphi_{(\epsilon)}$ belongs to $\mathcal{S}(\mathbb{R}^2)$ and vanishes at the origin, by Lemma 4.1 we have

$$\begin{aligned}
&\langle \eta, \hat{\varphi}_{(\epsilon)}\rangle = \int_{\mathbb{R}^2}\left(-\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\hat{\varphi}_{(\epsilon)}(x)\, dx + \frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x|\hat{\varphi}_{(\epsilon)}(x)\, dx\right)\Phi(y)\, dy \\
&= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} (2\pi|\xi|)^{-2}\varphi_{(\epsilon)}(\xi)(e^{-2\pi i\langle y,\xi\rangle} - 1)\, d\xi\right)\Phi(y)\, dy \\
&= \int_{\mathbb{R}^2} (2\pi|\xi|)^{-2}\varphi_{(\epsilon)}(\xi)(\hat{\Phi}(\xi) - 1)\, d\xi \\
&= \int_{\mathbb{R}^2} (2\pi|\xi|)^{-2}\varphi(\xi)(\hat{\Phi}(\xi) - 1)\, d\xi - \varphi(0)\int_{\mathbb{R}^2}(2\pi|\xi|)^{-2}\Psi(\xi/\epsilon)(\hat{\Phi}(\xi) - 1)\, d\xi.
\end{aligned}$$

Since $\Phi \in \mathcal{M}^1(\mathbb{R}^2)$, we can see that the last integral tends to 0 as $\epsilon \to 0$. Also, $\langle \eta, \hat{\varphi}_{(\epsilon)}\rangle = \langle \eta, \hat{\varphi}\rangle - \varphi(0)\langle \eta, (\hat{\Psi})_{\epsilon^{-1}}\rangle$ and $\langle \eta, (\hat{\Psi})_{\epsilon^{-1}}\rangle \to 0$ as $\epsilon \to 0$. Collecting results we get

$$\langle \eta, \hat{\varphi}\rangle = \int_{\mathbb{R}^2} (2\pi|\xi|)^{-2}\varphi(\xi)(\hat{\Phi}(\xi) - 1)\, d\xi,$$

which implies $\hat{\eta}(\xi) = (2\pi|\xi|)^{-2}(\hat{\Phi}(\xi) - 1)$. □

Let

$$\psi(x) = \Phi * L_2(x) - L_2(x) + c_0\Phi(x), \tag{4.4}$$

where $\Phi \in \mathcal{M}^1(\mathbb{R}^2)$ satisfying (1.7) and (1.8) and $c_0 = b_0/2$. Then, by the proof of Theorem 1.1 for $n \geq 3$ and Lemma 4.2, we can see that ψ satisfies (1.1) and (1), (2), (3) of Theorem A. Thus we have the following.

Theorem 4.3. *Let ψ be as in* (4.4)*. Suppose the condition* (1.2) *holds. Then*

$$\|f\|_{p,w} \simeq \|g_\psi(f)\|_{p,w}, \quad f \in L^p_w(\mathbb{R}^2).$$

If ψ is as in (4.4), then by Lemma 4.2 we see that $S_2(g) = g_\psi(H_0(g))$ for $g \in \mathcal{S}(\mathbb{R}^2)$. Using this and Theorem 4.3, we can argue similarly to the proof of Theorem 1.4 for $n \geq 3$, so that we see that Theorem 1.4 holds in the case of $\mathbb{R}^2$.

Also, Theorem B implies the following.

Theorem 4.4. *Let ψ be as in* (4.4)*. Suppose the conditions* (1.11) *and* (1.3) *hold. Then*

$$\|f\|_{p,w} \simeq \|\Delta_\psi(f)\|_{p,w}, \quad f \in L^p_w(\mathbb{R}^2).$$

Lemma 4.2 implies that $V_2(g) = \Delta_\psi(H_0(g))$, $g \in \mathcal{S}(\mathbb{R}^2)$. From this and Theorem 4.4 we can see that Theorem 1.5 is valid in the case of $\mathbb{R}^2$ by arguing similarly to the proof of Theorem 1.5 for $n \geq 3$.

5. One dimensional case

We recall the following result (see [5]).

Lemma 5.1. *Let $1 < \alpha \leq 2$, $\varphi \in \mathcal{S}(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} |x|^{\alpha-1}\hat{\varphi}(x)\,dx = \frac{1-\alpha}{2}\pi^{-\alpha+1/2}\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{3-\alpha}{2}\right)}\int_0^\infty \frac{\varphi(\xi)+\varphi(-\xi)-2\varphi(0)}{\xi^\alpha}\,d\xi.$$

We give a proof for completeness.

Proof of Lemma 5.1. We prove the lemma when $1 < \alpha < 2$. The case $\alpha = 2$ follows from this by taking the limit as $\alpha \to 2$ with $\alpha < 2$.

We write

$$\int_{-\infty}^{\infty} |x|^{\alpha-1}\hat{\varphi}(x)\,dx = \lim_{M\to\infty}\int_{-M}^{M} |x|^{\alpha-1}\hat{\varphi}(x)\,dx. \tag{5.1}$$

Now, integration by parts implies

$$\begin{aligned}\int_{-M}^{M} |x|^{\alpha-1}e^{-2\pi i\langle x,\xi\rangle}\,dx &= 2\int_0^M x^{\alpha-1}\cos(2\pi x\xi)\,dx \\ &= \int_0^M \Theta(\xi,x,M)(\alpha-1)x^{\alpha-2}\,dx,\end{aligned}$$

where

$$\Theta(\xi,x,M) = \frac{\sin(2\pi M\xi)}{\pi\xi} - \frac{\sin(2\pi x\xi)}{\pi\xi}.$$

Thus

$$\begin{aligned}\int_{-M}^{M} |x|^{\alpha-1}\hat{\varphi}(x)\,dx &= \int_0^\infty\int_0^M \Theta(\xi,x,M)(\varphi(\xi)+\varphi(-\xi))(\alpha-1)x^{\alpha-2}\,dx\,d\xi \\ &= \lim_{L\to\infty}\int_0^L\int_0^M \Theta(\xi,x,M)(\varphi(\xi)+\varphi(-\xi))(\alpha-1)x^{\alpha-2}\,dx\,d\xi.\end{aligned}$$

Let $\Psi(\xi) = \varphi(\xi) + \varphi(-\xi) - 2\varphi(0)$. Then we have

$$\int_0^L \int_0^M \Theta(\xi, x, M)(\varphi(\xi) + \varphi(-\xi))x^{\alpha-2}\,dx\,d\xi$$
$$= \int_0^L \int_0^M \Theta(\xi, x, M)\Psi(\xi)x^{\alpha-2}\,dx\,d\xi + 2\varphi(0)\int_0^L \int_0^M \Theta(\xi, x, M)x^{\alpha-2}\,dx\,d\xi.$$

We easily see that the last integral tends to 0 as $L \to \infty$, since

$$\int_0^L \frac{\sin(2\pi A\xi)}{\xi}\,d\xi \to \frac{\pi}{2} \quad \text{boundedly in } A > 0.$$

Therefore

$$\int_{-M}^M |x|^{\alpha-1}\hat{\varphi}(x)\,dx = \lim_{L\to\infty}\int_0^L \int_0^M \Theta(\xi, x, M)\Psi(\xi)(\alpha-1)x^{\alpha-2}\,dx\,d\xi. \tag{5.2}$$

By integration,

$$\int_0^L \int_0^M \frac{\sin(2\pi M\xi)}{\pi\xi}\Psi(\xi)(\alpha-1)x^{\alpha-2}\,dx\,d\xi = M^{\alpha-1}\int_0^L \frac{\sin(2\pi M\xi)}{\pi\xi}\Psi(\xi)\,d\xi.$$

Applying integration by parts, we have

$$M^{\alpha-1}\int_0^L \frac{\sin(2\pi M\xi)}{\pi\xi}\Psi(\xi)\,d\xi$$
$$= -2^{-1}\pi^{-2}M^{\alpha-2}\cos(2\pi ML)\Psi(L)/L + 2^{-1}\pi^{-2}M^{\alpha-2}\int_0^L \cos(2\pi M\xi)(\Psi(\xi)/\xi)'\,d\xi.$$

We observe that $(\Psi(\xi)/\xi)' \in L^1(\mathbb{R})$. Thus

$$\lim_{L\to\infty}\int_0^L \int_0^M \frac{\sin(2\pi M\xi)}{\pi\xi}\Psi(\xi)(\alpha-1)x^{\alpha-2}\,dx\,d\xi = 2^{-1}\pi^{-2}M^{\alpha-2}\int_0^\infty \cos(2\pi M\xi)(\Psi(\xi)/\xi)'\,d\xi. \tag{5.3}$$

We note that the last integral tends to 0 as $M \to \infty$. On the other hand, since $\Psi(\xi)\xi^{-\alpha}$ is integrable on the interval $(0, \infty)$, by a change of variables we have

$$\lim_{L\to\infty}\int_0^L \int_0^M \frac{\sin(2\pi x\xi)}{\pi\xi}\Psi(\xi)(\alpha-1)x^{\alpha-2}\,dx\,d\xi = \int_0^\infty \frac{\Psi(\xi)}{\pi\xi^\alpha}\int_0^{M\xi}(\alpha-1)x^{\alpha-2}\sin(2\pi x)\,dx\,d\xi. \tag{5.4}$$

Here we note that the limit

$$\lim_{M\to\infty}\int_0^M (\alpha-1)x^{\alpha-2}\sin(2\pi x)\,dx$$

exists when $1 < \alpha < 2$. By (5.2), (5.3) and (5.4), we see that

$$\lim_{M\to\infty}\int_{-M}^M |x|^{\alpha-1}\hat{\varphi}(x)\,dx = -(\alpha-1)2^{-\alpha+1}\pi^{-\alpha}\int_0^\infty x^{\alpha-2}\sin x\,dx\int_0^\infty \frac{\Psi(\xi)}{\xi^\alpha}\,d\xi.$$

By (5.1) and a formula for the value of the integral $\int_0^\infty x^{\alpha-2}\sin x\,dx$ (see [14, p. 182]), we get the conclusion. □

Remark 5.2. We note that

$$\frac{1-\alpha}{2}\pi^{-\alpha+1/2}\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{3-\alpha}{2}\right)} = 2(2\pi)^{-\alpha}\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right)$$

in Lemma 5.1.

We can prove the following.

Lemma 5.3. *Let* $L_2(x) = -\frac{1}{2}|x|$ *on* $\mathbb{R}^1$*. Suppose* $\Phi \in \mathcal{M}^1(\mathbb{R}^1)$ *and* $\operatorname{supp}\Phi \subset \{|x| \le M\}$*. Let* $\eta(x) = L_2 * \Phi(x) - L_2(x)$*. Then* $|\eta(x)| \le C$ *if* $|x| \le 2M$ *and* $\eta(x) = 0$ *if* $|x| \ge 2M$*. Also,* $\hat{\eta}(\xi) = (2\pi|\xi|)^{-2}(\hat{\Phi}(\xi) - 1)$*.*

The equation $\hat{\eta}(\xi) = (2\pi|\xi|)^{-2}(\hat{\Phi}(\xi) - 1)$ follows from Lemma 5.1 with $\alpha = 2$ as in Lemma 4.2. The other assertions of Lemma 5.3 can be shown easily.

Let

$$\psi(x) = \Phi * L_2(x) - L_2(x) + c_0\Phi(x), \tag{5.5}$$

where $\Phi \in \mathcal{M}^1(\mathbb{R}^1)$ and $c_0 = b_0/2$ with b_0 as in (1.7). Then, the conditions (1.1) and (1), (2), (3) of Theorem A follow from the proof of Theorem 1.1 for $n \ge 3$ and Lemma 5.3.

We have the following.

Theorem 5.4. *Let* ψ *be as in* (5.5)*. Then*

$$\|f\|_{p,w} \simeq \|g_\psi(f)\|_{p,w}, \quad f \in L^p_w(\mathbb{R}).$$

To see this from Theorem A, it suffices to show that (1.3) holds for ψ of (5.5). The proof is similar to the one given in Section 1 when Φ is a radial function. So, it suffices to show that ψ is not identically 0. We prove it by contradiction. Suppose that ψ is identically 0. Then,

$$\hat{\Phi}(\xi)(1 + c_0(2\pi|\xi|)^2) = 1.$$

Since $\hat{\Phi}$ is bounded and is not a constant function, we deduce that $c_0 > 0$. It follows that

$$\hat{\Phi}((2\pi)^{-1}c_0^{-1/2}\xi) = \frac{1}{1+\xi^2},$$

which is the Fourier transform of the function $\pi e^{-2\pi|x|}$. This contradicts the fact that Φ is compactly supported.

Let ψ be as in (5.5). Then it follows by Lemma 5.3 that $S_2(g) = g_\psi(H_0(g))$ for $g \in \mathcal{S}(\mathbb{R})$. Thus we can see that Theorem 1.4 holds in the case of $\mathbb{R}^1$ by applying the relation $S_2(g) = g_\psi(H_0(g))$ and Theorem 5.4 if we argue similarly to the proof of Theorem 1.4 for $n \ge 3$.

Also, by Theorem B we have the following.

Theorem 5.5. *Let* ψ *be as in* (5.5)*. Suppose the condition* (1.11) *holds. Then*

$$\|f\|_{p,w} \simeq \|\Delta_\psi(f)\|_{p,w}, \quad f \in L^p_w(\mathbb{R}).$$

By Lemma 5.3 we have $V_2(g) = \Delta_\psi(H_0(g))$, $g \in \mathcal{S}(\mathbb{R})$. Applying this and Theorem 5.5 and arguing similarly to the proof of Theorem 1.5 for $n \ge 3$, we can see that Theorem 1.5 holds on $\mathbb{R}^1$.

Remark 5.6. When $n = 1$, we do not need to assume the conditions (1.2) and (1.3) in Theorems 1.4 and 1.5, respectively, since they follow from the other hypotheses of the theorems, as we have seen above.

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