

On the local energy decay of higher derivatives of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in  $R^3$

メタデータ	言語: eng 出版者: 公開日: 2017-10-03 キーワード (Ja): キーワード (En): 作成者: メールアドレス: 所属:
URL	<a href="http://hdl.handle.net/2297/14520">http://hdl.handle.net/2297/14520</a>

# On the Local Energy Decay of Higher Derivatives of Solutions for the Equations of Motion of Compressible Viscous and Heat-conductive Gases in an Exterior Domain in $\mathbf{R}^3$

By Takayuki KOBAYASHI

Institute of Mathematics, University of Tsukuba

(Communicated by Kiyosi ITÔ, M. J. A., Sept. 12, 1997)

**1. Introduction.** Let  $\Omega$  be an exterior domain in  $\mathbf{R}^3$  with compact smooth boundary  $\partial\Omega$ . We consider the following system

$$(1.1) \quad \begin{cases} \rho_t + \gamma \operatorname{div} v = 0 & \text{in } [0, \infty) \times \Omega, \\ v_t - \alpha \Delta v - \beta \nabla (\operatorname{div} v) + \gamma \nabla \rho \\ \quad + \omega \nabla \theta = 0 & \text{in } [0, \infty) \times \Omega, \\ \theta_t - \kappa \Delta \theta + \omega \operatorname{div} v = 0 & \text{in } [0, \infty) \times \Omega, \\ v|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0 & \text{on } [0, \infty) \times \partial\Omega \\ (\rho, v, \theta)(0, x) = (\rho_0, v_0, \theta_0)(x) & \text{in } \Omega, \end{cases}$$

where  $\rho$  is the density,  $v = {}^T(v_1, v_2, v_3)$  the velocity and  $\theta$  the absolute temperature,  $\alpha, \gamma, \kappa$ , and  $\omega$  are positive numbers and  $\beta$  is a non-negative number. This system is the linearized equation of motion of compressible viscous and heat-conductive gases in an exterior domain in  $\mathbf{R}^3$ , which was given by Matsumura and Nishida [6] and Ponce [9]. Concerning the nonlinear problem, the unique existence of smooth solutions globally in time near constant state  $(\bar{\rho}_0, 0, \bar{\theta}_0)$  was studied by Matsumura and Nishida [8]. Deckelnick [2,3] proved the decay estimates for the solutions of nonlinear problem although the decay rate is weaker than that of Cauchy problem given by Matsumura and Nishida [6,7] and Ponce [9]. Our purpose is to get the decay estimates corresponding to Cauchy problem in the case of an exterior domain, which will be discussed in the forthcoming paper [5]. In our strategy, 1st step is to get local energy decay for the solutions of linearized equations (1.1). Kobayashi [4] proved the local energy decay of lower order derivatives of solutions. But since this system (1.1) is hyperbolic-parabolic type and since the regularity of solutions seems to be governed by the hyperbolic part  $\rho$ , we shall need to prove the regularity of solutions. Therefore in this paper we discuss a local energy decay estimates for higher order derivatives of solutions for the linearized

equations.

Now we shall state the main results. Let  $1 < q < \infty$ ,  $m$  be an integer and set

$\mathbf{X}_q^m(\Omega) = \{ {}^T U : U \in W_q^{m+1}(\Omega) \times \mathbf{W}_q^m(\Omega) \times W_q^m(\Omega) \}$ ,  $\mathbf{X}_q(\Omega) = \mathbf{X}_q^0(\Omega)$  where  ${}^T U$  means the transposed  $U$ ,  $W_q^m(\Omega) = \{ u \in L_q(\Omega) : \|u\|_{m,q,\Omega} = (\sum_{|\alpha| \leq m} \int_{\Omega} |\partial_x^\alpha u|^q dx)^{1/q} < \infty \}$  denotes the usual Sobolev spaces and  $\mathbf{W}_q^m(\Omega) = \{ \mathbf{W}_q^m(\Omega) \}^3$ . Define the  $5 \times 5$  matrix operator  $\mathbf{A}$  by the relation:

$$\mathbf{A} = \begin{pmatrix} 0 & \gamma \operatorname{div} & 0 \\ \gamma \nabla & -\alpha \Delta - \beta \nabla \operatorname{div} & \omega \nabla \\ 0 & \omega \operatorname{div} & -\kappa \Delta \end{pmatrix}$$

with the domain:

$$\mathcal{D}(\mathbf{A}) = \{ {}^T U = (\rho, v, \theta) \in W_q^1(\Omega) \times \mathbf{W}_q^2(\Omega) \times W_q^2(\Omega) : v|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega \}.$$

Let  $\mathbf{P}$  be the projection from  $\mathcal{D}(\mathbf{A})$  into  $\{ {}^T(v, \theta) \in \mathbf{W}_q^2(\Omega) \times W_q^2(\Omega) : v|_{\partial\Omega} = 0, \theta|_{\partial\Omega} = 0 \text{ on } \partial\Omega \}$ . Then by Kobayashi [4],  $-\mathbf{A}$  is a closed linear operator in  $\mathbf{X}_q(\Omega)$  and the resolvent set contain  $\Sigma = \{ \lambda \in \mathbf{C} : C \operatorname{Re} \lambda + (\operatorname{Im} \lambda)^2 > 0 \}$  where  $C$  is a constant depending only on  $\alpha, \beta, \gamma, \kappa$ , and  $\omega$ . Moreover, the following properties are valid; There exist positive constants  $\lambda_0$  and  $\delta < \frac{\pi}{2}$  such that

$$(1.2) \quad \|\lambda\| \|(\lambda + \mathbf{A})^{-1} \mathbf{F}\|_{\mathbf{X}_q(\Omega)} + \|\mathbf{P}(\lambda + \mathbf{A})^{-1} \mathbf{F}\|_{2,q,\Omega} \leq C(\lambda_0, \delta, m) \|\mathbf{F}\|_{\mathbf{X}_q(\Omega)}$$

for any  $\lambda - \lambda_0 \in \Sigma_\delta = \{ \lambda \in \mathbf{C} : |\arg \lambda| \leq \pi - \delta \}$  and any  $\mathbf{F} \in \mathbf{X}_q(\Omega)$ . This estimates means that  $-\mathbf{A}$  generates an analytic semigroup  $e^{-t\mathbf{A}}$  on  $\mathbf{X}_q(\Omega)$ .

Let  $b$  be a positive number such that  $\partial\Omega \subset B_b = \{ x \in \mathbf{R}^3 : |x| < b \}$ . Set  $\mathbf{Y}_{q,b}^m(\Omega) = \{ U = {}^T(\rho, v, \theta) \in \mathbf{X}_q^m(\Omega) : U(x) = 0 \text{ for } x \in \mathbf{R}^3 \setminus B_b, \int_{\Omega_b} \rho(x) dx = 0 \}$ , and  $\mathbf{Y}_{q,b}(\Omega) = \mathbf{Y}_{q,b}^0(\Omega)$  where  $\Omega_b = B_b \cap \Omega$ . Then

**Theorem 1.1.** *Let  $1 < q < \infty$  and let  $b_0$  be a fixed number such that  $B_{b_0} \supset \mathbf{R}^3 \setminus \Omega$ . Suppose that  $b > b_0$ . Then the following estimates are valid; for  $M \geq 0$  integers,  $\mathbf{U} \in \mathbf{Y}_{q,b}^1(\Omega)$  and  $t \geq 1$*

$$\|\partial_t^M e^{-tA} \mathbf{U}\|_{X_q^1(\Omega_b)} + \|\partial_t^M \mathbf{P} e^{-tA} \mathbf{U}\|_{L_{3,q,\Omega_b}} \leq C(q, b, M) t^{-3/2-M} \|\mathbf{U}\|_{X_q^1(\Omega_b)}.$$

**2. Proof of Theorem 1.1.** First we consider the stationary linearized equation with complex parameter  $\lambda$

$$(2.1) \quad (\lambda + \mathbf{A})\mathbf{U} = \mathbf{F} \text{ in } \Omega, \quad \mathbf{P}\mathbf{U} = 0 \text{ on } \partial\Omega.$$

**Lemma 2.1.** *Let  $1 < q < \infty$ . Then for  $\mathbf{F} \in X_q^1(\Omega)$  and  $\lambda - \lambda_0 \in \Sigma_\delta$*

$$|\lambda|^{-1/2} \|\mathbf{P}(\lambda + \mathbf{A})^{-1} \mathbf{F}\|_{L_{3,q,\Omega}} + |\lambda|^{1/2} \|(1 - \mathbf{P})(\lambda + \mathbf{A})^{-1} \mathbf{F}\|_{L_{2,q,\Omega}} \leq C \|\mathbf{F}\|_{X_q^1(\Omega)}.$$

*Proof.* First note that it follows from (1.2) and interpolation theorem that

$$(2.2) \quad |\lambda|^{1/2} \|(\lambda + \mathbf{A})^{-1} \mathbf{F}\|_{L_{1,q,\Omega}} \leq C \|\mathbf{F}\|_{X_q(\Omega)}$$

for  $\mathbf{F} \in X_q(\Omega)$  and  $\lambda - \lambda_0 \in \Sigma_\delta$ . Let  $\mathbf{U} = \mathbf{T}(\rho, v, \theta)$ ,  $\mathbf{F} = \mathbf{T}(f_1, f_2, f_3)$ . Applying the elliptic estimates to the system  $-\kappa \Delta$  and  $-\alpha \Delta - \beta \nabla \text{div}$  in (2.1) it follows from (2.2) and (1.2) that

$$\begin{aligned} \|v\|_{L_{3,q,\Omega}} &\leq C \{ |\lambda|^{1/2} \|\mathbf{F}\|_{X_q(\Omega)} + \|\mathbf{F}\|_{X_q^1(\Omega)} \\ &\quad + |\lambda|^{-1} \|v\|_{L_{3,q,\Omega}} \}, \\ \|\theta\|_{L_{3,q,\Omega}} &\leq C \{ |\lambda|^{1/2} \|\mathbf{F}\|_{X_q(\Omega)} + \|\mathbf{F}\|_{X_q^1(\Omega)} \}, \\ \|\rho\|_{L_{2,q,\Omega}} &\leq C \{ |\lambda|^{-1} \{ \|f_1\|_{L_{2,q,\Omega}} + \|v\|_{L_{3,q,\Omega}} \} \}. \end{aligned}$$

Taking  $\lambda_0$  sufficient large implies this Lemma by these estimates. ■

The following Lemma is concerned with low frequency of resolvent  $(\lambda + \mathbf{A})^{-1}$  near  $\lambda = 0$ . Let  $X$  and  $Y$  be Banach spaces,  $\mathcal{B}(X, Y)$  the set of all bounded linear operators from  $X$  into  $Y$  and  $\mathcal{A}(I; X)$  the set of all  $X$ -valued holomorphic functions in  $I$ . Then

**Lemma 2.2.** *Let  $1 < q < \infty$ ,  $b_0$  be a number such that  $B_{b_0} \subset \mathbf{R}^3 \setminus \Omega$  and let  $b > b_0$ . Put  $\mathcal{Y} = \mathcal{B}(Y_{q,b}(\Omega); \mathcal{D}(\mathbf{A}))$ . Then, there exist positive number  $\epsilon$  and  $\mathbf{R}(\lambda) \in \mathcal{A}(D_\epsilon; \mathcal{Y})$  where  $D_\epsilon = \{ \lambda \in \mathbf{C}; \text{Re} \lambda \geq 0, 0 < |\lambda| \leq \epsilon \}$  such that  $\mathbf{R}(\lambda)\mathbf{F} = (\lambda + \mathbf{A})^{-1}\mathbf{F}$ ,*

$$\begin{aligned} &\|(\frac{d}{d\lambda})^k \mathbf{R}(\lambda)\mathbf{F}\|_{X_q^m(\Omega_b)} + \|(\frac{d}{d\lambda})^k \mathbf{P}\mathbf{R}(\lambda)\mathbf{F}\|_{L_{2+m,q,\Omega_b}} \\ &\leq C(q, b, k, \epsilon, m) \max\{1, |\lambda|^{1/2-k}\} \|\mathbf{F}\|_{X_q^m(\Omega_b)}, \end{aligned}$$

for any  $\lambda \in D_\epsilon$ ,  $\mathbf{F} \in Y_{q,b}^m(\Omega)$  and  $k, m \geq 0$  integers.

*Proof.* The results for the case  $m = 0$  were proved by Kobayashi [4]. When  $m \geq 1$ , we can prove by employing the same argument as in Kobayashi [4]. In fact, we shall investigate the

parametrix which was constructed in [4]. First we consider the following stationary equations in  $\mathbf{R}^3$  with a complex parameter  $\lambda$

$$(2.3) \quad (\lambda + \mathbf{A})\mathbf{U} = \mathbf{F} \text{ in } \mathbf{R}^3.$$

By taking Fourier transform on (2.3) we obtain  $[\lambda + \hat{\mathbf{A}}(\xi)]\hat{\mathbf{U}} = \hat{\mathbf{F}}$ , where  $\mathcal{F}(f) = \hat{f}$  stand for the Fourier transforms of  $f$ . Here  $\hat{\mathbf{A}}$  is the  $5 \times 5$  symmetric matrix as follows:

$$\hat{\mathbf{A}}(\xi) = \begin{pmatrix} 0 & i\gamma\xi_k & 0 \\ i\gamma\xi_j & \delta_{jk}\alpha|\xi|^2 + \beta\xi_j\xi_k & i\omega\xi_j \\ 0 & i\omega\xi_k & \kappa|\xi|^2 \end{pmatrix}$$

where  $i = \sqrt{-1}$  and  $\delta_{jk} = 0$  when  $k \neq j$  and  $= 1$  when  $k = j$ . Set for  $\mathbf{F} \in X_q(\mathbf{R}^3)$

$$(2.4) \quad \mathbf{R}_0(\lambda)\mathbf{F}(x) = \mathbf{T}(\mathbf{R}_{0,\rho}(\lambda)\mathbf{F}(x), \mathbf{R}_{0,v}(\lambda)\mathbf{F}(x), \mathbf{R}_{0,\theta}(\lambda)\mathbf{F}(x)) = \mathcal{F}^{-1}\{[\lambda + \hat{\mathbf{A}}(\xi)]^{-1}\hat{\mathbf{F}}(\xi)\}(x).$$

Then we have the following estimates: Let  $1 < q < \infty$ ,  $b$  be a positive number. Then for  $\forall \mathbf{F} \in X_q^m(\mathbf{R}^3)$  with  $\mathbf{F}(x) = 0$  for  $x \in \mathbf{R}^3 \setminus B_b$  and  $\forall \lambda \in D_\epsilon$

$$(2.5) \quad \|(\frac{d}{d\lambda})^k \mathbf{R}_0(\lambda)\mathbf{F}\|_{X_q^m(B_b)} + \|(\frac{d}{d\lambda})^k \mathbf{P}\mathbf{R}_0(\lambda)\mathbf{F}\|_{L_{2+m,q,B_b}} \leq C \max\{1, |\lambda|^{1/2-k}\} \|\mathbf{F}\|_{X_q^m(\mathbf{R}^3)},$$

where  $k, m \geq 0$  are integers and  $C = C(\epsilon, q, b, k, m)$  is a constant. Moreover, for  $0 < \delta < 1/2$  and  $\lambda \in D_\epsilon$

$$(2.6) \quad \|\mathbf{T}\mathbf{R}_0(\lambda)\mathbf{F} - \mathbf{T}\mathbf{R}_0(0)\mathbf{F}\|_{W_q^{m+1}(B_b) \times W_q^{m+2}(B_b) \times W_q^{m+2}(B_b)} \leq C(\epsilon, \delta, q, m, b) |\lambda|^\delta \|\mathbf{F}\|_{X_q^m(\mathbf{R}^3)}.$$

In fact, since  $\partial_x^\alpha \partial_x^\beta \{\mathbf{R}_{0,v}(\lambda), \mathbf{R}_{0,\theta}(\lambda)\} \mathbf{F} = \partial_x^\alpha \{\mathbf{R}_{0,v}(\lambda), \mathbf{R}_{0,\theta}(\lambda)\} \partial_x^\beta \mathbf{F}$  where  $|\alpha| \leq 2, |\beta| \leq m$  and since  $\partial_x^\alpha \partial_x^\beta \mathbf{R}_{0,\rho}(\lambda)\mathbf{F} = \partial_x^\alpha \mathbf{R}_{0,\rho}(\lambda) \partial_x^\beta \mathbf{F}$  where  $|\alpha| \leq 1, |\beta| \leq m$ , it follows from the estimates (2.5) and (2.6) with  $m = 0$  which were proved by Kobayashi [4] that the estimates (2.5) and (2.6) with  $m \geq 1$  hold.

Next, let  $\mathbf{G} \in Y_{q,b}^m(\Omega)$ , and let  $\mathbf{W} \in W_q^{m+1}(\Omega_b) \times W_q^{m+2}(\Omega_b) \times W_q^{m+2}(\Omega_b)$  be the solution to the problem

$$\mathbf{A}\mathbf{W} = \mathbf{G} \text{ in } \Omega_b, \quad \mathbf{P}\mathbf{W} = 0 \text{ on } \partial\Omega_b.$$

The existence of such  $\mathbf{W}$  is guaranteed by Cattabriga [1]. In terms of  $\mathbf{W}$ , let us define the operator  $\mathbf{L}(0)$  by the relations:

$$\mathbf{W} = \mathbf{L}(0)\mathbf{G} = \{L_\rho(0)\mathbf{G}, L_v(0)\mathbf{G}, L_\theta(0)\mathbf{G}\}.$$

Here, note that by Cattabriga [1] we have the following estimates for any  $\mathbf{G} \in Y_{q,b}^m(\Omega)$

$$(2.7) \quad \|\mathbf{L}(0)\mathbf{G}\|_{X_q^m(\Omega_b)} + \|\mathbf{P}\mathbf{L}(0)\mathbf{G}\|_{L_{m+2,q,\Omega_b}} \leq C(q, b) \|\mathbf{G}\|_{X_q^m(\Omega_b)},$$

and  $L_\rho(0)\mathbf{G}$  is unique up to an additive constant.

Now, let  $b$  be a fixed constant  $b > R_0 + 3$ . Choosing  $\varphi$  in  $C^\infty(\mathbf{R}^3)$  so that  $\varphi(x) = 1$  for  $|x| \geq b - 1$  and  $= 0$  if  $|x| \leq b - 2$  and choosing  $\phi \in C_0^\infty(\Omega_b)$  so that  $\int_{\Omega_b} \phi(x) dx = 1$ , define the operator  $\mathbf{R}_1(\lambda)$  and  $\mathbf{S}(\lambda)$  by the relations: For  $\mathbf{F} \in \mathbf{Y}_{q,b}^m(\Omega)$  and  $\lambda \in D_\epsilon \cup \{0\}$

$$(2.8) \quad \mathbf{R}_1(\lambda)\mathbf{F} = \varphi\mathbf{R}_0(\lambda)\mathbf{F}_0 + (1 - \varphi)\mathbf{L}(0)\mathbf{F} - \frac{1}{\lambda} \int_{\Omega_b} S(\lambda)\mathbf{F} dx \varphi^T(1, 0, 0, 0, 0),$$

$$\mathbf{S}(\lambda)\mathbf{F} = {}^T\{S_\rho(\lambda)\mathbf{F}, S_v(\lambda)\mathbf{F}, S_\theta(\lambda)\mathbf{F}\},$$

where  $\mathbf{F}_0(x) = \mathbf{F}(x)$  for  $x \in \Omega$  and  $= 0$  for  $x \in \mathbf{R}^3 \setminus \Omega$ .

$$S(\lambda)\mathbf{F} = \lambda(1 - \varphi)L_\rho(0)\mathbf{F} + \gamma \nabla \varphi [\mathbf{R}_{0,\nu}(\lambda)\mathbf{F}_0 - \mathbf{L}_\nu(0)\mathbf{F}],$$

$$S_\rho(\lambda)\mathbf{F} = S(\lambda)\mathbf{F} - \int_{\Omega_b} S(\lambda)\mathbf{F} dx \phi,$$

$$S_\nu(\lambda)\mathbf{F} = \lambda(1 - \varphi)\mathbf{L}_\nu(0)\mathbf{F} - \alpha[\Delta\varphi + 2(\partial_j\varphi)\partial_j] [\mathbf{R}_{0,\nu}(\lambda)\mathbf{F}_0 - \mathbf{L}_\nu(0)\mathbf{F}]$$

$$- \beta \nabla \{ \partial_j \varphi [\mathbf{R}_{0,\nu}(\lambda)\mathbf{F}_0 - \mathbf{L}_\nu(0)\mathbf{F}] \}$$

$$- \beta \nabla \varphi \{ \text{div} [\mathbf{R}_{0,\nu}(\lambda)\mathbf{F}_0 - \mathbf{L}_\nu(0)\mathbf{F}] \}$$

$$+ \gamma \nabla \varphi [\mathbf{R}_{0,\rho}(\lambda)\mathbf{F}_0 - L_\rho(0)\mathbf{F}] + \omega \partial_j \varphi$$

$$[\mathbf{R}_{0,\theta}(\lambda)\mathbf{F}_0 - L_\theta(0)\mathbf{F}]_j - \frac{\gamma}{\lambda} \int_{\Omega_b} S(\lambda)\mathbf{F} dx \phi,$$

$$S_\theta(\lambda)\mathbf{F} = \lambda(1 - \varphi)L_\theta(0)\mathbf{F} - \kappa[\Delta\varphi + 2\partial_j\varphi\partial_j] [\mathbf{R}_{0,\theta}(\lambda)\mathbf{F}_0 - L_\theta(0)\mathbf{F}]$$

$$+ \omega \partial_j \varphi [\mathbf{R}_{0,\nu}(\lambda)\mathbf{F}_0 - \mathbf{L}_\nu(0)\mathbf{F}]_j.$$

Since  $L_\rho(0)\mathbf{F}$  is unique up to additive constant, we may choose  $L_\rho(0)\mathbf{F}$  in such a way that

$$(2.9) \quad \int_{\Omega_b} (1 - \varphi)L_\rho(0)\mathbf{F} dx = \int_{B_b} R_{0,\rho}(0)\mathbf{F}_0 dx - \int_{\Omega_b} \varphi R_{0,\rho}(0)\mathbf{F}_0 dx.$$

Note that the Stokes formula and (2.9) implies that

$$\int_{\Omega_b} S(\lambda)\mathbf{F} dx$$

$$= \lambda \int_{\Omega_b} (1 - \varphi)L_\rho(0) dx \mathbf{F} + \int_{B_b} \gamma \text{div} \mathbf{R}_{0,\nu}(\lambda)\mathbf{F}_0 dx - \int_{\Omega_b} \varphi \gamma \text{div} [\mathbf{R}_{0,\nu}(\lambda)\mathbf{F}_0 - \mathbf{L}_\nu(0)\mathbf{F}] dx$$

$$= \lambda \{ \int_{\Omega_b} (1 - \varphi)L_\rho(0)\mathbf{F} dx - \int_{B_b} R_{0,\rho}(\lambda)\mathbf{F}_0 dx + \int_{\Omega_b} \varphi R_{0,\rho}(\lambda)\mathbf{F}_0 dx \}.$$

It follows from (2.4), (2.5), (2.6), (2.7), (2.8), and (2.9) that

$$(2.10) \quad \mathbf{R}_1(\lambda) \in \mathcal{A}(D_\epsilon; \mathcal{Y}), \quad {}^T\mathbf{R}_1(0) \in \mathcal{B}(\mathbf{Y}_{q,b}^m(\Omega), W_{q,loc}^{m+1}(\Omega) \times W_{q,loc}^{m+2}(\Omega) \times W_{q,loc}^{m+2}(\Omega)),$$

$$(\lambda + \mathbf{A})\mathbf{R}_1(\lambda)\mathbf{F} = (1 + \mathbf{S}(\lambda))\mathbf{F} \text{ in } \Omega, \quad \mathbf{P}\mathbf{R}_1(\lambda)\mathbf{F} = 0 \text{ on } \partial\Omega,$$

$$\mathbf{S}(0) \in \mathcal{B}(\mathbf{Y}_{q,b}^m(\Omega), \mathbf{X}_q^{m+1}(\Omega)), \quad \mathbf{S}(\lambda) \in \mathcal{B}(\mathbf{Y}_{q,b}^m(\Omega), \{\mathbf{W}_q^{m+1}(\Omega)\}^5) \text{ for any } \lambda \in D_\epsilon.$$

Also we have  $\int_{\Omega_b} S_\rho(\lambda)\mathbf{F} dx = 0$  for  $\lambda \in D_\epsilon \cup \{0\}$  and

(2.11)  $\|\mathbf{S}(\lambda) - \mathbf{S}(0)\|_{\mathcal{B}(\mathbf{Y}_{q,b}^m(\Omega), \mathbf{Y}_{q,b}^m(\Omega))} \leq C(q, b, \delta)|\lambda|^\delta$  for  $\lambda \in D_\epsilon$  where  $0 < \delta < 1/2$ . Noting that  $\text{supp } \mathbf{S}(0)\mathbf{F}$  is contained in  $\Omega_b$ , it follows from (2.11) and Rellich's compactness theorem that  $\mathbf{S}(0)$  is a compact operator from  $\mathbf{Y}_{q,b}^1(\Omega)$  into itself. Since  $1 + \mathbf{S}(0)$  is injective in  $\mathcal{B}(\mathbf{Y}_{q,b}(\Omega), \mathbf{Y}_{q,b}(\Omega))$  by Lemma 4.6 in Kobayashi [4], by Fredholm's alternative theorem,  $1 + \mathbf{S}(0) \in \mathcal{B}(\mathbf{Y}_{q,b}^m(\Omega), \mathbf{Y}_{q,b}^m(\Omega))$  has the bounded inverse  $(1 + \mathbf{S}(0))^{-1}$ . Thus putting  $\|(1 + \mathbf{S}(0))^{-1}\|_{\mathcal{B}(\mathbf{Y}_{q,b}^m(\Omega), \mathbf{Y}_{q,b}^m(\Omega))} = M$ , by (2.11), there exists an  $\epsilon > 0$  such that  $1 + \mathbf{S}(\lambda)$  also has the bounded inverse  $(1 + \mathbf{S}(\lambda))^{-1}$  from  $\mathbf{Y}_{q,b}^m(\Omega)$  onto itself whenever  $\lambda \in D_\epsilon$ , and moreover

$$(2.12) \quad \|(1 + \mathbf{S}(\lambda))^{-1}\|_{\mathcal{B}(\mathbf{Y}_{q,b}^m(\Omega), \mathbf{Y}_{q,b}^m(\Omega))} \leq 2M \text{ for } \lambda \in D_\epsilon.$$

It follows from (2.5), (2.7), (2.8), and (2.10) that for  $\mathbf{F} \in \mathbf{Y}_{q,b}^m(\Omega)$ ,  $\lambda \in D_\epsilon$  and  $k \geq 0$  integer

$$(2.13) \quad \|(\frac{d}{d\lambda})^k \mathbf{R}_1(\lambda)\mathbf{F}\|_{\mathbf{X}_q^m(\Omega_b)} + \|(\frac{d}{d\lambda})^k \mathbf{P}\mathbf{R}_1(\lambda)\mathbf{F}\|_{m+2,q,\Omega_b} \leq C \max\{1, |\lambda|^{1/2-k}\} \|\mathbf{F}\|_{\mathbf{X}_q^m(\Omega_b)}.$$

Thus putting  $\mathbf{R}(\lambda) = \mathbf{R}_1(\lambda)(1 + \mathbf{S}(\lambda))^{-1}$ , combining (2.12) and (2.13) implies Lemma 2.2. ■

Now we shall prove our main theorem. To do this we prepare the following lemma, which was proved by Shibata (see Theorems 3.2 and 3.7 of [10]).

**Lemma 2.3.** *Let  $X$  be a Banach space with norm  $|\cdot|_X$ . Let  $f(\tau)$  be a function of  $C^\infty(\mathbf{R} \setminus \{0\}; X)$  such that  $f(\tau) = 0$ ,  $|\tau| \geq a$  with some  $a > 0$ . Assume that there exists a constant  $C(f)$  depending on  $f$  such that for any  $0 < |\tau| \leq a$ ,*

$$|(\frac{d}{d\tau})^k f(\tau)|_X \leq C(f)|\tau|^{-1/2-k}, \quad k = 0, 1.$$

Put  $g(t) = \int_{-\infty}^\infty f(\tau)e^{-i\tau t} d\tau$ . Then

$$|g(t)|_X \leq C(1 + |t|)^{-1/2} C(f).$$

Let  $\mathbf{U} \in \mathbf{Y}_{q,b}^1(\Omega)$ ,  $b > b_0$  and let  $\phi \in C_0^\infty(\mathbf{R}^3)$  such that  $\phi(x) = 1$  for  $|x| \leq b$  and  $= 0$  for  $|x| \geq b + 1$ . Taking  $\eta(s) \in C^\infty(\mathbf{R})$  so that  $\eta(s) = 1$  for  $|s| \leq 1/4$  and  $= 0$  for  $|s| \geq 1/2$  we can represent the semigroup as follows (see Kobayashi [4]):

$$(2.14) \quad \phi e^{-i\mathbf{A}t} \mathbf{U} = \mathbf{J}_0(t)\mathbf{U} + \mathbf{J}_\infty(t)\mathbf{U}$$

where

$$\mathbf{J}_0(t)\mathbf{U} = \frac{1}{2\pi i} \left( \phi \int_{-\infty}^\infty e^{its} \eta(s) \frac{d}{ds} (is + \mathbf{A})^{-1} \mathbf{U} ds \right),$$

$$\mathbf{J}_\infty(t) \mathbf{U} = \frac{1}{2\pi t} \left( \phi \int_{-\infty}^{\infty} e^{its} (1 - \eta(s)) \frac{d}{ds} (is + \mathbf{A})^{-1} \mathbf{U} ds \right).$$

By (1.2), (2.2), and by Lemma 2.1 we have

$$\begin{aligned} & \|D_x^\alpha (1 - \eta(s)) \left(\frac{d}{ds}\right)^N (is + \mathbf{A})^{-1} \mathbf{U}\|_{q,\Omega} \\ (2.15) \quad & \leq (1 - \eta(s)) \{ \| (is + \mathbf{A})^{-N-1} \mathbf{U}\|_{X_{1,q}^1(\Omega)} \\ & \quad + \| \mathbf{P}(is + \mathbf{A})^{-N-1} \mathbf{U}\|_{3,q,\Omega} \} \\ & \leq C(N) (1 + |S|)^{-(N-1)/2} \| \mathbf{U}\|_{X_{1,q}^1(\Omega)}, \end{aligned}$$

where  $D_x^\alpha = (\partial_x^{\alpha_1}, \dots, \partial_x^{\alpha_5})$ ,  $|\alpha_1| \leq 2$ ,  $|\alpha_j| \leq 3$  ( $j = 2, \dots, 5$ ) and hence by the relation  $\frac{1}{t} \cdot \frac{d}{d\lambda} e^{t\lambda} = e^{t\lambda}$ , we have

(2.16)  $\|D_x^\alpha \partial_t^M \mathbf{J}_\infty(t) \mathbf{U}\|_{q,\Omega} \leq C(N, M, \alpha) t^{-N} \| \mathbf{U}\|_{X_{1,q}^1(\Omega)}$   
 for any integers  $N \geq 2$ ,  $M \geq 0$ . On the other hand, noting that

$$\begin{aligned} D_x^\alpha \partial_t^M \mathbf{J}_0(t) \mathbf{U} &= \frac{1}{2\pi} \sum_{n=0}^M \binom{M}{N} \partial_t^{M-N} t^{-1} D_x^\alpha \\ & \quad \left\{ \phi \int_{-\infty}^{\infty} e^{its} \eta(s) (is)^n \frac{d}{ds} \mathbf{R}(is) \mathbf{U} ds \right\} \end{aligned}$$

it follows from Lemma 2.2 and Lemma 2.3 that

$$(2.17) \quad \|D_x^\alpha \partial_t^M \mathbf{J}_0(t) \mathbf{U}\|_{q,\Omega} \leq C(M, b, q) (1+t)^{-(M+3/2)} \| \mathbf{U}\|_{X_{1,q}^1(\Omega)}$$

for any  $\mathbf{U} \in \mathbf{Y}_{q,b}^1(\Omega)$ , integer  $M \geq 0$  and  $t \geq 1$ . Combining (2.15), (2.16), and (2.17) implies Theorem 1.1. This completes the proof.

### References

[ 1 ] L. Cattabriga: Su un problema al contorno relativo al sistema di equazioni di Stokes. Rend. Mat. Sem. Univ. Padova., **31**, 308–340 (1961).

[ 2 ] K. Deckelnick: Decay estimates for the compressible Navier-Stokes equations in unbounded domain. Math. Z., **209**, 115–130 (1992).  
 [ 3 ] K. Deckelnick:  $L^2$ -decay for the compressible Navier-Stokes equations in unbounded domains. Commun. in partial Differential Equations, **18**, 1445–1476 (1993).  
 [ 4 ] T. Kobayashi: On a local energy decay of solutions for the equations of motion of viscous and heat-conductive gases in an exterior domain in  $\mathbf{R}^3$ . Tsukuba J. Math. (to appear).  
 [ 5 ] T. Kobayashi and Y. Shibata: Decay estimates of solutions for the equation of motion of compressible viscous and heat-conductive gases in an exterior domain in  $\mathbf{R}^3$  (preprint).  
 [ 6 ] A. Matsumura and T. Nishida: The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. Proc. Japan Acad., **55A**, 337–342 (1979).  
 [ 7 ] A. Matsumura and T. Nishida: The unitial value problems for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ., **20**(1), 67–104 (1980).  
 [ 8 ] A. Matsumura and T. Nishida: Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. Commun. Math. Phys., **89**, 445–464 (1983).  
 [ 9 ] G. Ponce: Global existence of small solutions to a class of nonlinear evolution equations. Nonlinear. Anal. TM., **9**, 339–418 (1989).  
 [ 10 ] Y. Shibata: On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain. Tsukuba J. Math., **7**, 1–68 (1983).